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WELFARE OPTIMAL INFORMATION STRUCTURES IN PUBLIC GOOD PROVISION

CHRISTOPH SCHOTTMÜLLER

Department of Economics
University of Cologne
Albertus-Magnus-Platz
D-50923 Köln
Germany

http://www.wiso.uni-koeln.de

Welfare optimal information structures in public good provision

Christoph Schottmüller*

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Abstract

This paper studies welfare maximizing information structures in a public good setting. In large groups less information is provided. In the limit, no information is provided but the information is efficient as by the law of large numbers no information is needed to take the efficient decision. This implies that the free rider problem is most severe not for large but for intermediate group sizes.

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1. Introduction

Free riding has been identified as a main obstacle to efficiency in the decision of whether to provide a public good or not. That is, agents that derive a high benefit from the public good are tempted to contribute very little in the hope that the public good will be provided anyway due to more generous contributions of others. Conventional wisdom suggests that this problem is more severe if the population is large as the contribution of any individual is less likely to be pivotal. Indeed even under optimal contribution mechanisms the free riding problem can become so severe that the public good is never provided if the number of agents is large.[1]

This paper considers not only the optimal mechanism but also the optimal information structure. Which information is revealed before the decision about a public project is taken will determine how precisely agents can judge their own expected benefit. Providing detailed information allows agents to precisely know their own benefit and has therefore the advantage that – in principle – it is possible to determine whether the sum of benefits is higher than the costs, i.e. whether a provision of the public good is efficient or not. The downside of precise information are increased "information rents" because each agent's benefit from the project is eventually *privately* known by the agent. More precise information lead therefore to higher information rents and those imply less efficient decisions.

This paper makes the point that for large populations of agents with independent benefits from the project, the free rider problem does not exist if the use of information design is possible. The reason is that – by the law of large numbers – the aggregate benefit of the project

*University of Cologne, Faculty of Management, Economics and Social Sciences, Albertus Magnus Platz, 50923 Köln, Germany; Tilec and C-SEB; email: c.schottmueller@uni-koeln.de.

[1] I will prove a version of this result in lemma 1.

can be estimated relatively precisely if the number of agents is large even if no information is provided, i.e. the average benefit will equal the mean of the prior distribution of individual benefits. Consequently, providing no information and therefore reducing information rents to 0 is (approximately) optimal if the population is large and allows to take the efficient decision.

It follows immediately that the free riding problem is most pronounced not for large but for intermediate group sizes: for very small group sizes free riding is only a minor problem even without information design and for large group sizes information design eliminates the free riding problem.

This paper establishes the above mentioned results and also characterizes welfare maximizing, symmetric information structures in a public good model.[2] In particular, it is shown that optimal information structures are monotone partitions of the type space and therefore have finite support if the type space is finite. To illustrate the results, an explicit solution is given for the case of a binary type space.

The related literature on information design has recently been surveyed in Bergemann and Morris (2019). Methodologically closest are Bergemann and Pesendorfer (2007) who derive the revenue maximizing information structure in a model of optimal auctions and Schottmüller (2022) who analyzes the welfare maximizing information structure in a model of bilateral trade. The optimal information structure is a monotone partition of the type space in both papers and similar methods of proof are used to derive this result. The public good model without information design is, of course, well known and many textbooks cover the derivation of the optimal mechanism in this setup; see Börgers (2015, ch. 3.3) for a textbook exposition and d'Aspremont and Gérard-Varet (1979) for an early contribution.

2. Model

There are I agents where $I \geq 2$. The agents have to choose whether to provide a public good or not. This decision is denoted by $y \in [0,1]$ where y is the probability with which the public good is provided. Agent $i \in \{1, \ldots, I\}$ has valuation v_i for the public good. I assume that v_i is distributed on $[\underline{v}, \bar{v}]$ according to a distribution with cumulative distribution function H and that agents' valuations are independent. Put differently, valuations are independently and identically distributed across agents. The expected value of H is denoted by μ and its variance by $\sigma^2 > 0$. Agent i's payoff equals $yv_i - t_i$ where t_i is a transfer payment agent i may have to make (either to finance the public good or to compensate other agents). The costs of the public good are denoted by c. There is no source of outside financing and therefore $\sum_{i=1}^{I} t_i \geq c$ if $y = 1$ and $\sum_{i=1}^{I} t_i \geq 0$ if $y = 0$.

Agents know neither their own valuation v_i nor the valuation of any other agent $j \neq i$. However, each agent observes a noisy signal θ_i of his own valuation (and this signal is

[2] Symmetry means in this context that all agents receive an independent signal from the same information structure. There are several reasons for the restriction to symmetry. First, it is well known that with correlated signals first best can be achieved generically following arguments in Cremer and McLean (1988). Second, this case seems to be of particular interest for democratic societies in which equal, non-discriminatory treatment of agents is required. Third, it is natural to interpret an information structure as a a set of attributes that is made publicly available and then compared by the agents with their personal needs. As the actual information is public, this would immediately imply symmetry of the information structure.

independent of other players' signals and valuations). As agents care only about expected valuations it is without loss of generality to identify a signal with the expected valuation it induces. Therefore, the signal technology is denoted by a distribution F of expected valuations. That is, the support of F, denoted by Θ, is a subset of $[\underline{v}, \bar{v}]$ and by Bayes' consistency H has to be a mean preserving spread of F. Note that all players are assumed to have access to the same information technology and therefore draw their types θ_i independently from the same distribution F, see footnote 2 for a justification of this assumption.

Given an information structure F, it is – by the revelation principle – without loss of generality to consider only incentive compatible direct revelation mechanisms. A direct revelation mechanism consists of two functions $q : \Theta^I \to [0,1]$ and $t : \Theta^I \to \mathbb{R}^I$ where $q(\theta)$ is the probability that the public good is provided if the vector of types is $\theta = (\theta_1, \dots, \theta_I)$ and $t(\theta) = (t_1(\theta), \dots, t_I(\theta))$ is the vector of transfers agents have to pay given type vector θ. A direct revelation mechanism is incentive compatible if

$$\mathbb{E}_{\theta_{-i}}\left[y(\theta)\theta_i - t_i(\theta)\right] \geq \mathbb{E}_{\theta_{-i}}\left[y(\tilde{\theta}_i, \theta_{-i})\theta_i - t_i(\tilde{\theta}_i, \theta_{-i})\right] \qquad \forall i \in \{1, \dots, I\},\ \theta_i \in \Theta,\ \tilde{\theta}_i \in \Theta. \quad (1)$$

Participation in the mechanism is assumed to be voluntary at the interim stage. That is, the participation constraint

$$\mathbb{E}_{\theta_{-i}}\left[y(\theta)\theta_i - t_i(\theta)\right] \geq 0 \quad (2)$$

has to hold for all $i \in \{1, \dots, I\}$ and $\theta_i \in \Theta$.

The main objective of the paper is to find the information structure F and mechanism (y, t) that jointly maximize expected welfare from an ex ante point of view. Expected welfare equals $\sum_{i=1}^{I} \mathbb{E}_\theta\left[y(\theta)\theta_i - t_i(\theta)\right]$, i.e. equal welfare weights for all agents are assumed. It is furthermore assumed that the vector of valuations is relevant for the welfare optimal decision, i.e. $I\underline{v} < c < \bar{v}I$. Note that in those section dealing with limit results in I costs are allowed to vary in I and are therefore written as $c(I)$.[3]

3. Analysis

3.1. Limit result

The following result states that expected welfare in the optimal information structure converges to first best welfare if the number of agents grows large. Put differently, the free rider problem that hampers efficiency without information design can be eradicated by information design. The intuition is that, by the law of large numbers, the average valuation becomes predictable as the number of agents grows large. In fact, it is unnecessary to provide the agents with any information in the limit as I tends to infinity as all the necessary information for an efficient decision is contained in the prior. This is reminiscent of decision making behind the "veil of ignorance": the decision is made in a state in which payoff consequences for specific individuals are unknown.

Proposition 1. *Let $c(I)$ be the cost of the public good as a function of the number of agents I*

[3] A classic example for a public good is street lighting. It makes intuitive sense that the costs of street lighting are higher for cities with more inhabitants as those cities are likely to consist of more streets.

and assume that $\bar{c} \equiv \lim_{I\to\infty} c(I)/I$ exists. If $\bar{c} \neq \mu$, then expected welfare under the optimal information structure equals expected first best welfare in the limit as $I \to \infty$.

Proof of proposition 1: Denote the limit $\lim_{I\to\infty} c(I)/I$ as $\bar{c}$ and recall that μ (σ) denotes the expected value (standard deviation) of H. I distinguish two cases.

Case 1: $\mu > \bar{c}$ Expected welfare in the optimal information structure is bounded from below by the welfare of the totally uninformative information structure, i.e. every agent receives the same signal independent of valuation, and the mechanism that provides the public good regardless of signals while allocating an equal cost share to each agent. Note that by $\mu > \bar{c}$ this mechanism is incentive compatible and satisfies the participation constraint in the totally uninformative information structure.

The expected welfare loss of always providing the public good compared to first best can be written as $L = \int_{-\infty}^{c(I)} c(I) - W \, dG(W)$ where $W = \sum_{i=1}^{I} v_i$ and $G(W)$ denotes the distribution of W. The central limit theorem states that $Z = (\sum_{i=1}^{I}(v_i - \mu))/(\sqrt{I}\sigma)$ is (approximately) standard normally distributed if I is sufficiently large. Denoting the cdf (pdf) of the standard normal distribution as Φ (ϕ), L can for large I therefore be rewritten as

$$\begin{aligned} L &\approx \int_{-\infty}^{\frac{c(I)-I\mu}{\sqrt{I}\sigma}} c(I) - I\mu - Z\sqrt{I}\sigma \, d\Phi(Z) = -\Phi\left(\frac{c(I)-I\mu}{\sqrt{I}\sigma}\right)\left(\mathbb{E}\left[Z|Z \leq \frac{c(I)-I\mu}{\sqrt{I}\sigma}\right]\sqrt{I}\sigma - c(I) + I\mu\right) \\ &= -\Phi\left(\sqrt{I}\frac{c(I)/I-\mu}{\sigma}\right)\left(-\frac{\phi\left(\frac{c(I)-I\mu}{\sqrt{I}\sigma}\right)}{\Phi\left(\frac{c(I)-I\mu}{\sqrt{I}\sigma}\right)}\sqrt{I}\sigma - c(I) + I\mu\right) \\ &= \phi\left(-\sqrt{I}\frac{\mu - c(I)/I}{\sigma}\right)\sqrt{I}\sigma - \left(\mu - \frac{c(I)}{I}\right) I\Phi\left(-\sqrt{I}\frac{\mu - c(I)/I}{\sigma}\right) \end{aligned}$$

where the third step uses the well known fact that $\mathbb{E}[Z|Z \leq a] = -\phi(a)/\Phi(a)$ if Z is standard normally distributed. By assumption, $\mu - c(I)/I$ converges to the positive number $(\mu - \bar{c})/\sigma$ as I grows large. This implies that $\lim_{I\to\infty} \phi\left(-\sqrt{I}\frac{\mu-c(I)/I}{\sigma}\right)\sqrt{I}\sigma = 0$. Furthermore, the second term in the last line converges to zero as $\lim_{x\to\infty} x\Phi(-\sqrt{x}) = 0$. This proves the claim for case 1.

Case 2: $\mu < \bar{c}$ The proof is analogous to case 1 but using the mechanism that never provides the public good in combination with the completely uninformative information structure. □

Note that this result contrasts sharply with a conventional mechanism design result stating that welfare converges to zero if the number of agents grows large for a fixed information structure. That is, the free riding problem becomes maximal as the number of agents grows if a (somewhat informative) information structure is fixed but the free rider problem becomes irrelevant if the information structure is chosen optimally.

Lemma 1. *Let $c(I)$ be the cost of the public good as a function of the number of agents I and assume that $\bar{c} > \underline{\theta}$ exists where $\underline{\theta} = \inf supp(F)$ and $\bar{c} = \lim_{I\to\infty} c(I)/I$. Expected welfare under the optimal mechanism for a given information structure equals zero in the limit as $I \to \infty$.*

3.2. Welfare maximal mechanism for a given information structure

Before turning to the optimal information structure it is necessary to determine the optimal mechanism for a given finite information structure. The derivation is similar to the standard derivation with a continuum of types, see for example Börgers (2015, ch. 3.3.4).

Let $\Theta = \{\theta^1, \ldots, \theta^n\}$ and denote the probability that θ_i equals θ^k as f^k. In a given direct revelation mechanism denote by $Y_i(\theta^k)$ the expected probability that agent i with signal θ^k assigns to the public good being provided, i.e. $Y_i(\theta^k) = \mathbb{E}_{\theta_{-i}}[y(\theta)|\theta_i = \theta^k]$. Denote the expected utility of agent i with type θ^k in a given mechanism by $U_i(\theta^k) = Y_i(\theta^k)\theta^k - T_i(\theta^k)$ where the expected transfer $T_i(\theta^k)$ is defined as $T_i(\theta^k) = \mathbb{E}_{\theta_{-i}}[t_i(\theta)|\theta_i = \theta^k]$. The following lemma characterizes incentive compatibility.

Lemma 2. *A direct revelation mechanism is incentive compatible if and only if Y_i is increasing for all $i = 1, \ldots, I$ and*

$$U_i\left(\theta^k\right) = U_i\left(\theta^{k-1}\right) + \tilde{Y}_i\left(\theta^{k-1}\right)\left(\theta^k - \theta^{k-1}\right) \tag{3}$$

for some $\tilde{Y}_i(\theta^{k-1}) \in \left[Y_i(\theta^{k-1}), Y_i(\theta^k)\right]$.

The objective is to maximize expected welfare which equals

$$\mathbb{E}_\theta\left[y(\theta)\left(\sum_{i=1}^{I}\{\theta_i\} - c\right)\right] = \sum_{i=1}^{I}\sum_{k=1}^{n} Y_i(\theta^k) f^k\left(\theta^k - \frac{c}{I}\right). \tag{4}$$

The main constraint, next to incentive compatibility and participation constraints, is the budget balance constraint which can be written as[4]

$$\begin{aligned}
\mathbb{E}_\theta\left[\sum_{i=1}^{I}\{T_i(\theta_i)\} - y(\theta)c\right] &\geq 0 \\
\Leftrightarrow \sum_{i=1}^{I} \mathbb{E}_{\theta_i}\left[Y_i(\theta_i)\theta_i - U_i(\theta_i) - Y_i(\theta_i)c/I\right] &\geq 0
\end{aligned}$$

which after plugging in (3) becomes[5]

$$\begin{aligned}
\Leftrightarrow \sum_{i=1}^{I}\sum_{k=1}^{n} f^k\left[Y_i(\theta^k)\theta^k - U_i(\theta^1) - \sum_{m=1}^{k-1}\tilde{Y}_i(\theta^m)(\theta^{m+1} - \theta^m) - Y_i(\theta^k)c/I\right] &\geq 0 \\
\Leftrightarrow -\sum_{i=1}^{I}\{U_i(\theta^1)\} + \sum_{i=1}^{I}\sum_{k=1}^{n}\left[f^k Y_i(\theta^k)(\theta^k - c/I) - \tilde{Y}_i(\theta^k)(\theta^{k+1} - \theta^k)(1 - F(\theta^k))\right] &\geq 0.
\end{aligned}$$

By the participation constraint $U_i(\theta^1) \geq 0$. As $U_i(\theta^1)$ does not enter the objective, it is clearly optimal to choose it as low as possible in order to relax the budget balance constraint, i.e.

[4] It is well known that ex ante budget balance is equivalent to ex post budget balance in this setup, see Börgers and Norman (2009) or Proposition 3.6 in Börgers (2015, p. 48). I use the simpler to handle ex ante version here.

[5] I use the convention $\theta^{n+1} = \theta^n$ to simplify notation.

$U_i(\theta^1) = 0$. Furthermore, it is optimal to choose $\tilde{Y}_i$ as low as possible to relax the budget balance constraint. By lemma 2, $\tilde{Y}_i(\theta^m) \geq Y_i(\theta^m)$ and therefore $\tilde{Y}_i(\theta^m) = Y_i(\theta^m)$ is optimal. These considerations yield the implementation condition:

$$\sum_{i=1}^{I} \sum_{k=1}^{n} Y_i(\theta^k) \left(\theta^k - \frac{c}{I} - (\theta^{k+1} - \theta^k) \frac{1 - F(\theta^k)}{f^k} \right) f^k \geq 0. \tag{C}$$

Define δ_i as $\theta^{k+1} - \theta^k$ if $\theta_i = \theta^k$. Neglecting the monotonicity constraint for the time being, the Lagrangian is then

$$\begin{aligned} \mathcal{L} &= \sum_{i=1}^{I} \sum_{k=1}^{n} Y_i(\theta^k) \left(\left[\theta^k - \frac{c}{I} \right] (1 + \lambda) - \lambda(\theta^{k+1} - \theta^k) \frac{1 - F(\theta^k)}{f^k} \right) f^k \\ &= \mathbb{E}_\theta \left[y(\theta) \sum_{i=1}^{I} \left(\left[\theta_i - \frac{c}{I} \right] (1 + \lambda) - \lambda \delta_i \frac{1 - F(\theta_i)}{f(\theta_i)} \right) \right]. \end{aligned} \tag{5}$$

The optimal mechanism is then

$$y^*(\theta) \begin{cases} = 1 & \text{if } \sum_{i=1}^{I} \left[\theta_i - \frac{c}{I} \right] (1 + \lambda) - \lambda \delta_i \frac{1 - F(\theta_i)}{f(\theta_i)} > 0 \\ \in [0, 1] & \text{if } \sum_{i=1}^{I} \left[\theta_i - \frac{c}{I} \right] (1 + \lambda) - \lambda \delta_i \frac{1 - F(\theta_i)}{f(\theta_i)} = 0 \\ = 0 & \text{if } \sum_{i=1}^{I} \left[\theta_i - \frac{c}{I} \right] (1 + \lambda) - \lambda \delta_i \frac{1 - F(\theta_i)}{f(\theta_i)} < 0. \end{cases} \tag{6}$$

Note that y^* is symmetric across players and therefore I will write Y, U and T instead of Y_i, U_i and T_i in the following. However, it remains to check that the neglected monotonicity constraint on Y is indeed not binding in the solution. Note that this would be obvious if (C) was slack, i.e. if $\lambda = 0$. If $\lambda \neq 0$, monotonicity appears to be hard to verify on first sight as the distribution F will be the welfare maximizing information structure that is endogenous in the setting of this paper. However, the following lemma verifies that the monotonicity constraint does not bind in the welfare maximizing information structure with n signals.[6]

Lemma 3. *Let F be the welfare maximizing information structure with at most n signals. Then, the monotonicity constraint on Y does not bind.*

3.3. Optimal information structure

The following result states that welfare maximizing information structures have a relatively simple form. More specifically, they are monotone partitions of the type space. That is, if H has a density, then the optimal information structure is given by a number of cutoffs $(c^0, c^1, \ldots, c^n)$ such that all types in between c^{k-1} and c^k receive signal θ^k. If H is discrete or has mass points, it is more natural to think of cutoffs $(c^0 = 0, c^1, \ldots, c^{n-1}, c^n = 1)$ where $f^k = c^k - c^{k-1}$ and the lowest (highest) type receiving signal θ^k is given by $H^{-1}(c^k)$ $(H^{-1}(c^{k-1}))$ where H^{-1} is the pseudo-inverse of H.

[6] It is in fact straightforward to extend the lemma to optimal information structures with an infinite number of signals and therefore neglecting the monotonicity constraint in the derivation of the optimal mechanism is unproblematic given that F is the optimal information structure.

Proposition 2. *The welfare maximizing information structure with at most n signals is a monotone partition of the type space.*

One may wonder why the optimal information structure with at most n signals is of independent interest. The first reason is that a welfare level arbitrarily close to the optimal welfare level can be achieved by a finite information structure.

Lemma 4. *Take any information structure F and denote expected welfare in this information structure (using the optimal mechanism) by W_F. Then for any $\varepsilon > 0$ there exists an information structure F_n with finite support such that welfare under F_n (using the optimal mechanism) is at least $W_F - \varepsilon$.*

The second reason is that combining proposition 2 and lemma 4 implies that the optimal information structure is finite whenever the support of H is finite. The optimal information structure is therefore a finite, monotone partition of the type space if the distribution of valuations, H, has finite support.

Corollary 1. *The information structure maximizing expected welfare is finite if the support of H is finite.*

3.4. Example: binary types

This subsection deals with the special case in which the support of the true type distribution H is binary. Corollary 1 already established that in this case the optimal information structure is finite. The following lemma extends this result by establishing that the optimal information structure is in fact binary.

Lemma 5. *Let the support of H be binary. The support of the expected welfare maximizing information structure is then binary.*

Given a binary signal structure, the first best mechanism can be stated as follows: the public good should be procured if $\theta^h I^h + \theta^l I^l \geq c(I)$ where I^h (I^l) is the number of agents with a high signal θ^h (θ^l). That is, there is a cutoff $\hat{I}^h$ such that the public good is procured under the first best rule if and only if the number of agents with a high signal weakly exceeds $\hat{I}^h$.

The expected welfare maximizing second best mechanism in (6) can also be stated in simple terms. In the binary case, (6) can be rewritten as saying that the public good is provided only if

$$I^h\left(\theta^h - \frac{c(I)}{I}\right) + I^l\left(\theta^l - \frac{c(I)}{I}\right) \geq I^l \frac{\lambda}{1+\lambda}\frac{1-f^l}{f^l}(\theta^h - \theta^l).$$

The first best rule provides the public good whenever the left hand side is positive. That is, the public good is provided in less cases than under first best, i.e. only if the number of agents with a high signal is sufficiently above $\hat{I}^h$. Given this, it is clear that the optimal mechanism provides the public good if and only if the number of agents with a high signal is (weakly) above a threshold $\tilde{I}^h$. This threshold is chosen as the lowest I^h weakly above $\hat{I}^h$

such that (C) holds because choosing a higher threshold would clearly reduce welfare. To be more precise, the optimal mechanism might use mixing so that (C) holds with equality. For example, when choosing the threshold equal to 7 might lead to a budget deficit but choosing it equal to 8 might create a surplus. In this case, the public good is provided for sure if I^h is 8 or higher and is provided with a probability $\alpha > 0$ in case I^h equals 7. The probability α is chosen such that (C) holds with equality. For completeness, note that (C) with binary signals can be written as

$$Y(\theta^l)\left(\theta^l - \frac{c(I)}{I} - (\theta^h - \theta^l)\frac{1-f^l}{f^l}\right) f^l + Y(\theta^h)\left(\theta^h - \frac{c(I)}{I}\right)(1-f^l) \geq 0$$

where

$$\begin{aligned}
Y(\theta^l) &= \alpha\binom{I-1}{\tilde{I}^h}(1-f^l)^{\tilde{I}^h} f^{l\,I-1-\tilde{I}^h} + \sum_{k=\tilde{I}^h+1}^{I}\binom{I-1}{k}(1-f^l)^k f^{l\,I-1-k} \\
Y(\theta^h) &= \alpha\binom{I-1}{\tilde{I}^h-1}(1-f^l)^{\tilde{I}^h-1} f^{l\,I-\tilde{I}^h} + \sum_{k=\tilde{I}^h}^{I}\binom{I-1}{k}(1-f^l)^k f^{l\,I-1-k}.
\end{aligned}$$

By lemma 5, an information structure in the binary setting can be described by one number, e.g. the share f^l of agents receiving a low signal. With a binary signal technology, which is optimal by lemma 5, this yields immediately $f^h = 1-f^l$. Concentrating on monotone partitions, which are optimal by proposition 2, the signals immediately follow: if $f^l < h^l$, then $\theta^l = v^l$ and $\theta^h = v^l(h^l - f^l)/f^h + v^h h^h/f^h$; if $f^l \geq h^l$, then $\theta^h = v^h$ and $\theta^l = v^l h^l/f^l + v^h(f^l - h^l)/f^l$. For the information structure associated with f^l, the optimal mechanism can be computed as above. Consequently, the search for the optimal information structure can be written as a one-dimensional maximization problem over f^l. Although this problem has no closed form solution it is easily solvable numerically.

Figure 1 illustrates per agent welfare in the optimal information structure. Welfare under the optimal mechanism without designing the information structure tends – in line with lemma 1 – to zero as the number of agents increases. The reason for this is the well known free-rider problem. As the number of agents grows, first a gap between first best welfare and welfare in the optimal information structure emerges. This gap closes as the number of agents gets large as implied by proposition 1. The welfare gap is most problematic for intermediate number of agents because the free rider problem is already severe while information design cannot make use of the predictability of the large numbers yet.[7]

[7] One may wonder why first best welfare is decreasing in the number of agents. This is easiest explained in an example: if there is a single agent whose valuation is high or low, then he will buy the good if and only if the valuation is high and therefore the full upside of a high valuation is exploited. If there are many agents, then sometimes the first agent will have a high valuation but the other agents have low valuations. In these cases the public good may not be provided and the high valuation of the first agent is "wasted".

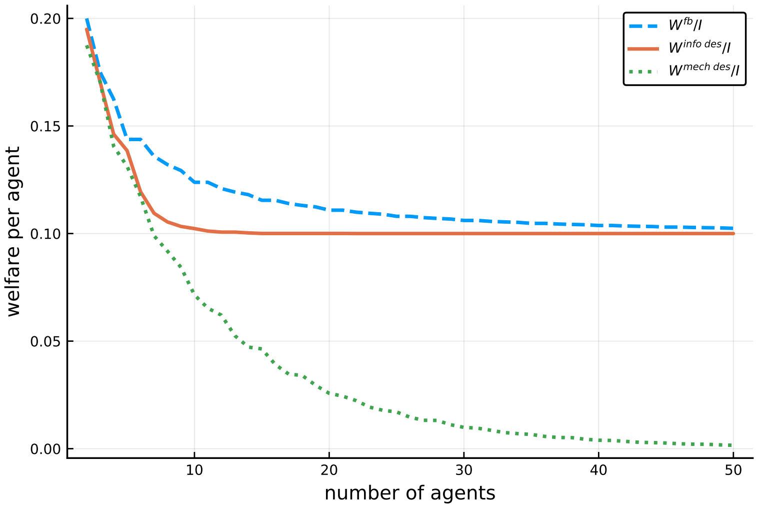

Figure 1: Per agent welfare as a function of I under first best (dashed), using the optimal information structure (solid) and using the optimal mechanism with a fully informative information structure (dotted). Parameters: $c(I) = 0.4I$, $v^h = 1$, $v^l = 0$, $h^l = 0.5$

4. Conclusion

While an increase in the number of agents exacerbates the free riding problem in standard public good problems and thereby makes efficiency impossible, the opposite is true if information design can be used. Namely, first best efficiency can be reached in the limit if the number of agents grows large. The idea is that the efficient decision can be made based upon the prior due to the law of large numbers. Hence, no information beyond the prior is optimally given to the agents. In practice, this could for example mean that the decision is taken without detailed inquiries into distributional consequences and at an early stage before payoff consequences to specific individuals are known. This is reminiscent of decision making behind the “veil of ignorance”, i.e. before any private information is realized.

Appendix

Proof of lemma 1: The proof is given for the case where F has a density f. However, this is for notational convenience only and the result is true for finite information structures as well.[8] To ease notation let $\bar{f}(\theta) = prod_{i=1}^{i} f(\theta_i)$, i.e. $\bar{f}(\theta)$ is the density of the vector $\theta = (\theta_1, \ldots, \theta_I)$. Following Börgers (2015, p. 55), the implementation condition can be written as

$$\int_{\Theta^I} y(\theta) \left[\sum_{i=1}^{I} \{ \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} - \frac{c(I)}{I} \} \right] \bar{f}(\theta) \, d\theta \geq 0.$$

Let $\tilde{\Theta} = \{\theta | y(\theta) = 0\}$. I will show that the implementation condition implies that $\tilde{\Theta}$ has full probability in the limit as $I \to \infty$. Given that the probability that the public good is provided is zero in the limit, expected welfare equals zero as $I \to \infty$.

Consider the random variable

$$Z_i = \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} - \bar{c}$$

where θ_i is distributed according to F. Then

$$\mathbb{E}[Z_i] = \mu - \int_{\Theta} 1 - F(\theta_i) \, d\theta_i - \bar{c} = \underline{\theta} - \bar{c} < 0$$

where the second equality uses integration by parts. Given that Z_i for $i = 1, \ldots, I$ are independently and identically distributed with finite expectation, $\sum_{i=1}^{I} Z_i / I$ converges almost surely to $\underline{\theta} - \bar{c} < 0$ as $I \to \infty$ by the law of large numbers, see for example Theorem 8.32 in Capinski and Kopp (2004, p. 266). Note that the implementation condition can be rewritten as

$$\int_{\Theta^I} y(\theta) I \left[\bar{c} - c(I)/I + \sum_{i=1}^{I} Z_i / I \right] \bar{f}(\theta) \; d\theta \geq 0.$$

The previous step implies that the term in brackets is strictly negative with probability 1 in the limit as $I \to \infty$. Consequently, the implementation condition requires $y(\theta) = 0$ with probability 1 in the limit as $I \to \infty$. □

Proof of lemma 2: *If:* Let (3) hold and Y_i be increasing. Let $k > j$. Iterating (3), yields

$$U_i(\theta^k) = U_i(\theta^j) + \sum_{m=j}^{k-1} \tilde{Y}_i(\theta^m)(\theta^{m+1} - \theta^m). \tag{7}$$

As $\tilde{Y}_i(\theta^m) \geq Y_i(\theta^m) \geq Y_i(\theta^j)$ by the monotonicity of Y_i, this implies

$$U_i(\theta^k) \geq U_i(\theta^j) + \sum_{m=j}^{k-1} Y_i(\theta^j)(\theta^{m+1} - \theta^m) = U_i(\theta^j) + Y_i(\theta^j)(\theta^k - \theta^j)$$

[8] The steps of the proof for the finite case are the same: One starts with the implementation condition (C) and utilizes the "discrete integration by parts" formula $\sum_{k=1}^{n} -(\theta^{k+1} - \theta^k)(1 - F(\theta^k)) = -\mu + \theta_1 < 0$. Everything else remains virtually unchanged. Mixed distributions can be approximated arbitrarily closely by finite distributions.

which is equivalent to saying that signal θ^k does not want to misrepresent as signal θ^j.

Similarly, (7) in combination with $\tilde{Y}_i(\theta^m) \leq \tilde{Y}_i(\theta^{m+1}) \leq Y_i(\theta^k)$ implies

$$U_i(\theta^k) \leq U_i(\theta^j) + \sum_{m=j}^{k-1} Y_i(\theta^k)(\theta^{m+1} - \theta^m) = U_i(\theta^j) + Y_i(\theta^k)(\theta^k - \theta^j)$$

which is equivalent to saying that signal θ^j does not want to misrepresent as signal θ^k. As i, j, k were arbitrary this proves that the conditions of the lemma imply incentive compatibility.

Only if: Let the mechanism be incentive compatible. This implies in particular that $U_i(\theta^k) \geq U_i(\theta^{k-1}) - Y_i(\theta^{k-1})\theta^{k-1} + Y_i(\theta^{k-1})\theta^k$ and also $U_i(\theta^{k-1}) \geq U_i(\theta^k) - Y_i(\theta^k)\theta^k + Y_i(\theta^k)\theta^{k-1}$. Taking these two inequalities together yields

$$Y_i(\theta^k) \geq \frac{U_i(\theta^k) - U_i(\theta^{k-1})}{\theta^k - \theta^{k-1}} \geq Y_i(\theta^{k-1})$$

which implies that $Y_i(\theta^k) \geq Y_i(\theta^{k-1})$, i.e. Y_i is increasing, and that (3) holds with $\tilde{Y}_i(\theta^{k-1}) = (U_i(\theta^k) - U_i(\theta^{k-1}))/(\theta^k - \theta^{k-1})$. □

Proof of lemma 3: The claim is obvious if $\lambda = 0$. Therefore, let $\lambda \neq 0$ in the following. Suppose contrary to the claim in the lemma that $Y(\theta^k) = Y(\theta^{k+1})$ in the optimal information structure with at most n signals and let k be the lowest bunched signal if more than two signals are bunched on the same Y. The proof idea is to "merge" the two signals θ^k and θ^{k+1} to one signal. It will be shown that this does not directly affect the objective but strictly relaxes (C) and therefore the desired contradiction is obtained. The intuitive idea underlying the proof is that merging signals leads to less information and therefore lower information rents.

Consider the alternative information structure that differs from the optimal one by merging θ^k and θ^{k+1} to signal $\tilde{\theta} = \theta^k f^k/(f^k + f^{k+1}) + \theta^{k+1} f^{k+1}/(f^k + f^{k+1})$ with $f(\tilde{\theta}) = f^k + f^{k+1}$. Adjust the decision rule y in case at least one agent has signal $\tilde{\theta}$ by taking the expected decision. E.g. if agent i has signal $\tilde{\theta}$ (and no other agent has signal $\tilde{\theta}$) then $y(\tilde{\theta}, \theta_{-i}) = y(\theta^k, \theta_{-i}) f^k/(f^k + f^{k+1}) + y(\theta^{k+1}, \theta_{-i}) f^{k+1}/(f^k + f^{k+1})$. Note that this leaves $Y(\theta^m)$ unchanged for $m \notin \{k, k+1\}$ and $Y(\tilde{\theta}) = Y(\theta^k) = Y(\theta^{k+1})$. This implies that the objective value in (4) is not affected by the merging of signals.

It remains to show that (C) is strictly relaxed. Focussing on the inner sum in (C), the only terms that are affected by the change can be written as[9]

$$\begin{aligned}
& Y(\theta^{k-1})\left(-\theta^k(1-F(\theta^{k-1}))\right)+Y_i(\theta^k)\left(\theta^k f^k-\frac{c}{I}f^k-(\theta^{k+1}-\theta^k)(1-F(\theta^k))\right)\\
& +Y(\theta^{k+1})\left(\theta^{k+1}f^{k+1}-\frac{c}{I}f^{k+1}-(\theta^{k+2}-\theta^{k+1})(1-F(\theta^{k+1}))\right)\\
= \;& Y(\theta^{k-1})\left(-\theta^k(1-F(\theta^{k-1}))\right)+\tilde{Y}\left(\tilde{\theta}\tilde{f}-\frac{c}{I}\tilde{f}-(\theta^{k+2}-\tilde{\theta})(1-F(\theta^{k+1})\right)\\
& +\tilde{Y}\left(-\tilde{\theta}(1-F(\theta^{k+1}))-f^{k+1}\theta^{k+1}+(1-F(\theta^k))\theta^k\right)\\
= \;& Y(\theta^{k-1})\left(-\tilde{\theta}(1-F(\theta^{k-1}))\right)+\tilde{Y}\left(\tilde{\theta}\tilde{f}-\frac{c}{I}\tilde{f}-(\theta^{k+2}-\tilde{\theta})(1-F(\theta^{k+1})\right)\\
& Y(\theta^{k-1})(1-F(\theta^{k-1}))(\tilde{\theta}-\theta^k)+\tilde{Y}\left(-\tilde{\theta}(1-F(\theta^{k+1}))-f^{k+1}\theta^{k+1}+(1-F(\theta^k))\theta^k\right)\\
< \;& Y(\theta^{k-1})\left(-\tilde{\theta}(1-F(\theta^{k-1}))\right)+\tilde{Y}\left(\tilde{\theta}\tilde{f}-\frac{c}{I}\tilde{f}-(\theta^{k+2}-\tilde{\theta})(1-F(\theta^{k+1})\right)\\
& \tilde{Y}(1-F(\theta^{k-1}))(\tilde{\theta}-\theta^k)+\tilde{Y}\left(-\tilde{\theta}(1-F(\theta^{k+1}))-f^{k+1}\theta^{k+1}+(1-F(\theta^k))\theta^k\right)\\
= \;& Y(\theta^{k-1})\left(-\tilde{\theta}(1-F(\theta^{k-1}))\right)+\tilde{Y}\left(\tilde{\theta}\tilde{f}-\frac{c}{I}\tilde{f}-(\theta^{k+2}-\tilde{\theta})(1-F(\theta^{k+1})\right)\\
& +\tilde{Y}\left(\tilde{\theta}\tilde{f}-f^{k+1}\theta^{k+1}-f^k\theta^k\right)\\
= \;& Y(\theta^{k-1})\left(-\tilde{\theta}(1-F(\theta^{k-1}))\right)+\tilde{Y}\left(\tilde{\theta}\tilde{f}-\frac{c}{I}\tilde{f}-(\theta^{k+2}-\tilde{\theta})(1-F(\theta^{k+1})\right).
\end{aligned}$$

As the last line contains exactly the relevant terms for (C) in the modified information structure, it follows that the merging of signals relaxed the binding constraint (C). □

Proof of proposition 2: Suppose otherwise. That is, there exist set of types N^k and N^{k+1} with mass $\eta > 0$ such that all types in N^k receive signal θ^k and all types in N^{k+1} receive signal θ^{k+1} while $\mathbb{E}[v_i|v_i \in N^k] > \mathbb{E}[v_i|v_i \in N^{k+1}]$.

Consider the optimization problem of maximizing (5) over θ^k and θ^{k+1} while fixing y, F and all other θ^j (for $j \neq, k, k+1$) at their welfare values in the welfare maximizing information structure and mechanism.[10] More specifically, I consider a one-dimensional optimization problem over a parameter ε where

$$\begin{aligned}
\theta^k(\varepsilon) &= \frac{(f^k-\varepsilon)\theta^k+\varepsilon\theta^{k+1}}{f^k}\\
\theta^{k+1}(\varepsilon) &= \frac{(f^{k+1}-\varepsilon)\theta^{k+1}+\varepsilon\theta^k}{f^{k+1}}.
\end{aligned}$$

Note that for $\varepsilon = 0$, $\theta^k(0) = \theta^k$ and $\theta^{k+1}(0) = \theta^{k+1}$. That is, optimality of θ^k and θ^{k+1} imply that $\mathcal{L}$ has to be maximized by $\varepsilon = 0$ over the set of ε that yield feasible signal distributions. I will argue below that an open neighborhood of 0 is feasible for ε in this sense

[9] To cut short on notation, I use $\tilde{Y} = Y(\theta^k) = Y(\theta^{k+1})$ and $\tilde{f} = f^k + f^{k+1}$.

[10] Welfare maximizing information structure and mechanism exist by the Weierstrass theorem as the problem is finite dimensional by assumption.

and therefore the derivative of $\mathcal{L}$ with respect to ε has to be zero at $\varepsilon = 0$:

$$\frac{\partial \mathcal{L}}{\partial \varepsilon} = I(\theta^{k+1} - \theta^k)\left[Y(\theta^k)(1+\lambda+\lambda(1-F^k)/f^k) - \lambda Y(\theta^{k-1})(1-F^{k-1})/f^k\right] \\ - I(\theta^{k+1} - \theta^k)\left[Y(\theta^{k+1})(1+\lambda+\lambda(1-F^{k+1})/f^{k+1}) - \lambda Y(\theta^k)(1-F^k)/f^{k+1}\right].$$

Note that ε is not part of $\partial \mathcal{L}/\partial \varepsilon$, i.e. $\mathcal{L}$ is linear in ε. That is, $\partial \mathcal{L}/\partial \varepsilon|_{\varepsilon=0} \neq 0$ unless $\mathcal{L}$ is linear with slope 0 in ε, i.e. unless ε does not affect $\mathcal{L}$. I will now argue that $\partial \mathcal{L}/\partial \varepsilon = 0$ contradicts the assumed optimality of y and θ^k and θ^{k+1}.

Suppose to the contrary that $\partial \mathcal{L}/\partial \varepsilon = 0$. There exists an $\varepsilon' > 0$ such that

$$\theta^k(\varepsilon') - \frac{c}{I} - (\theta^{k+1}(\varepsilon') - \theta^k(\varepsilon'))\frac{1-F^k}{f^k} = \theta^{k+1}(\varepsilon') - \frac{c}{I} - (\theta^{k+2} - \theta^{k+1}(\varepsilon'))\frac{1-F^{k+1}}{f^{k+1}},$$

i.e. an $\varepsilon' > 0$ such that the virtual valuation of $\theta^k(\varepsilon')$ equals the virtual valuation of signal $\theta^{k+1}(\varepsilon')$. This is true as (i) for $\varepsilon' = 0$, the virtual valuation of θ^{k+1} is higher than that of θ^k, (ii) the virtual valuation of θ^k is increasing in ε, (iii) the virtual valuation is decreasing in ε and (iv) the virtual valuation of $\theta^k(\varepsilon)$ would exceed the virtual valuation of $\theta^{k+1}(\varepsilon)$ if ε was so high that $\theta^k(\varepsilon) = \theta^{k+1}(\varepsilon)$. As $\partial \mathcal{L}/\partial \varepsilon = 0$ and $\mathcal{L}$ is linear in ε, $\mathcal{L}$ evaluated at ε' equals $\mathcal{L}$ evaluated at $\varepsilon = 0$. As a next step, change y by averaging over θ^k and θ^{k+1}, i.e. $\tilde{y}(\theta_i = \theta^k, \theta_{-i}) = f^k y(\theta_i = \theta^k, \theta_{-i})/(f^k + f^{k+1}) + f^{k+1} y(\theta_i = \theta^{k+1}, \theta_{-i})/(f^k + f^{k+1})$ and also $\tilde{y}(\theta_i = \theta^{k+1}, \theta_{-i}) = f^k y(\theta_i = \theta^k, \theta_{-i})/(f^k + f^{k+1}) + f^{k+1} y(\theta_i = \theta^{k+1}, \theta_{-i})/(f^k + f^{k+1})$.[11] As $\mathcal{L}$ is linear in y and the virtual valuation is the same for $\theta^k(\varepsilon')$ and $\theta^{k+1}(\varepsilon')$, this change in y does not affect the value of $\mathcal{L}$. Note, however, that $\tilde{Y}^k = \tilde{Y}^{k+1}$. This implies by the same argument as in the proof of lemma 3 that merging the two signals $\theta^k(\varepsilon')$ and $\theta^{k+1}(\varepsilon')$ will not affect the objective but strictly relax the binding implementation condition and therefore strictly increase $\mathcal{L}$. But this implies that $(\theta^k, \theta^{k+1}, y(\theta^k, \cdot), y(\theta^{k+1}, \cdot))$ do not jointly maximize $\mathcal{L}$ in an auxilliary problem in which all other variables are fixed at their optimal value. This, however, contradicts the optimality of $(\theta^k, \theta^{k+1}, y(\theta^k, \cdot), y(\theta^{k+1}, \cdot))$.

As it was now shown that $\partial \mathcal{L}/\partial \varepsilon = 0$ is impossible in the optimum, all that remains to be shown is that there exists an open neighborhood of ε around zero such that the information structures created above are feasible in this neighborhood, i.e. H is a mean preserving spread of the distributions generated by $(\theta^1, \dots, \theta^k(\varepsilon), \theta^{k+1}(\varepsilon), \dots, \theta^n)$ and $(f^1, \dots, f^n)$. Note that for $\varepsilon \geq 0$ this follows immediately from feasibility of the original information structure as an increase in ε "adds noise" to the original information structure by sending ε of those types that originally received signal θ^k the signal θ^{k+1} and vice versa. The feasibility for $\varepsilon < 0$ follows from the assumption that the original information structure is not a monotone partition. To see that $\varepsilon < 0$ is feasible, consider changing the information structure by swapping the signal of mass $\tau < \eta$ in N^k and N^{k+1}, i.e mass $\tau < \eta$ of the types in N^k receives signal θ^{k+1} (instead of θ^k) and mass τ in N^{k+1} receives signal θ^k (instead of θ^{k+1}). This is clearly feasible and does not change f^k or f^{k+1} but the expected valuation when receiving signals θ^k or θ^{k+1} changes

[11] It is understood that this transformation is applied iteratively if several players have a signal in $\{\theta^k, \theta^{k+1}\}$.

to

$$\tilde{\theta}^k(\tau) = \frac{f^k\theta^k - \tau\left(\mathbb{E}[v|v \in N^k] - \mathbb{E}[v|v \in N^{k+1}]\right)}{f^k}$$
$$\tilde{\theta}^{k+1}(\tau) = \frac{(\omega^{k+1}\theta^{k+1} + \tau\left(\mathbb{E}[v|v \in N^k] - \mathbb{E}[v|v \in N^{k+1}]\right)}{f^{k+1}}.$$

Choosing $\tau = -\varepsilon(\theta^{k+1} - \theta^k)/\left(\mathbb{E}[v|v \in N^k] - \mathbb{E}[v|v \in N^{k+1}]\right)$ yields $\theta^k(\varepsilon)$ and $\theta^{k+1}(\varepsilon)$ for negative ε. □

Proof of lemma 4: Consider the hypothetical problem of maximizing expected welfare subject to budget balance being violated by no more than η (through the choice of an information structure and mechanism). Denote the by $W^*(\eta)$ the value of this maximization problem (more formally, the supremum of welfare achievable by information structures and mechanisms that do not violate the ex ante budget balance constraint by more than η). As both expected welfare and the budget balance condition are continuous in mechanism and information structure, W^* is also continuous. Let $\tilde{\eta} < 0$ be such that $W^*(0) - W^*(\tilde{\eta}) < \varepsilon/3$. (Note that a negative η indicates a stricter constraint.)

Define the set of distributions $\mathcal{F}_\kappa$ as the set of distributions with cdfs F_κ such that (i) $\mathbb{E}_{F_\kappa}[v] \leq \mathbb{E}_H[v] - \kappa$ and (ii) $\int_{-\infty}^x F_\kappa(v)\, dv \leq \int_{-\infty}^x H(v+\kappa)\, dv - \kappa$ for all $x \in (-\infty, \max supp(H) - \kappa]$. Note that $\mathcal{F}_0$ is the feasible sets of distributions in the welfare maximization problem of this paper as the set of mean preserving spreads of a distribution equals the set of distributions that have the same mean while also second order stochastically dominating the distribution, see Mas-Colell et al. (1995, ch. 6.D).

Consider now the problem of maximizing welfare subject to budget balance being violated by no more than $\tilde{\eta}$ over the sets $\mathcal{F}_\kappa$. Let F denote an information structure such that under this information structure and the optimal mechanism (i) budget balance is violated by at most $\tilde{\eta}$, (ii) welfare is above $W^*(\tilde{\eta}) - \varepsilon/3$ and (iii) $F \in \mathcal{F}_{\tilde{\kappa}}$ for some $\tilde{\kappa} > 0$. Such F and $\tilde{\kappa}$ exist by the definition of $\tilde{\eta}$ and as the conditions defining $\mathcal{F}_\kappa$ are continuous in κ (while welfare and budget balance constraint are continuous in signals).[12]

Approximate F by a series of distributions $(F_n)_{n=1}^\infty$ such that (i) the support of F_n has at most n elements and (ii) $F_n \to F$ almost everywhere. Then F_n converges to F weakly and by the Helly-Bray theorem welfare and budget balance under F_n converge to the corresponding values under F.[13] Therefore for some sufficiently high n^* welfare under F_{n^*} is above $W^*(\tilde{\eta}) - 2\varepsilon/3 > W^*(0) - \varepsilon$ and budget balance is "violated" by at most $\tilde{\eta} < 0$. But this implies – by $\tilde{\eta} < 0$ – that under the finite information structure F_{n^*} welfare above $W^*(0) - \varepsilon$ is achievable without violating budget balance. Finally, define $F^*_{n^*}$ by "shifting F_{n^*} up" such that $F^*_{n^*}$ has expected value $\mathbb{E}_H[v]$, i.e. $F^*_{n^*}(x) = F_{n^*}\left(x - \mathbb{E}_H[v] + \mathbb{E}_{F_{n^*}}[v]\right)$ and note that the definition of $\mathcal{F}_{\tilde{\kappa}}$ implies $\mathbb{E}_H[v] - \mathbb{E}_{F_{n^*}}[v] > 0$ (for n^* sufficiently high). Note that shifting the distribution

[12]For readability and notational convenience, I assume in the following that F is continuous. If F already has finite support, the following approximation step is, of course, unnecessary. If F is mixed with a finite number of mass points, the following approximation is understood to be applied only to the continuous part, i.e. F_n has the same mass points as F but also discretizes its continuous parts.

[13]We use the same mechanism as under F here. For completeness, define $y(\theta) = sup_{\theta' \leq \theta,\, \theta' \in supp(F)} y(\theta')$ for all θ not in the support of F (and let $y(\theta) = 0$ if $y(\theta')$ is not defined for any $\theta' \leq \theta$). This ensures the monotonicity of Y.

of valuations up by a constant, increases welfare and relaxes the budget balance constraint. Consequently, welfare under $F^*_{n^*}$ is above $W(0)-\varepsilon$. Furthermore, H is a mean preserving spread of $F^*_{n^*}$ by the definition of $\mathcal{F}_{\tilde{\kappa}}$. Consequently, welfare of at least $W(0)-\varepsilon$ can be achieved by a feasible finite information structure. □

Proof of corollary 1: Let the support of H consist of m elements. Let W^* be the supremum of the set of welfare values achievable by all information structures and mechanism under constraint (C). For $n = 1, 2\ldots$, there exists a finite information structure that yields welfare W^*-1/n by lemma 4. By proposition 2, this information structure is a monotone partition. A monotone partition of H leads to a signal structure with at most $2m-1$ elements because the support of H consisted of only m elements. Put differently, for any $\varepsilon > 0$ welfare greater than $W^*-\varepsilon$ can be achieved by a monotone partition with at most $2m-1$ elements. Therefore, welfare of W^* can be achieved by a monotone partition with at most $2m-1$ elements. This shows that the welfare maximizing information structure is finite if the support of H is finite. □

Proof of lemma 5: This proof utilitzes a more general result stated here as a separate lemma:

Lemma 6. *Let the true type distribution of buyer valuations H be discrete and let its support be $\{\hat{v}_1, \hat{v}_2, \ldots\}$. If $\hat{v}_i$ and $\hat{v}_{i+1}$ are in the support of the optimal signal distribution with at most n signals, then the optimal information structure assigns zero probability to all signals in $(\hat{v}_i, \hat{v}_{i+1})$.*

Proof of lemma 6: Suppose otherwise, i.e. let the optimal information structure put positive probability on types $\theta^{-i} < \theta^i < \theta^{i+1}$ and let θ^{i-1} and θ^{i+1} be neighboring elements in the support of H.[14] Denote the corresponding probabilities in the optimal information structure by f^{i-1}, f^i and f^{i+1}. We will consider the following alternative distributions indexed by ε:

$$\begin{aligned}
\tilde{f}^{i-1}(\varepsilon) &= f^{i-1} - \varepsilon\frac{\theta^{i+1}-\theta^i}{\theta^{i+1}-\theta^{i-1}}\\
\tilde{f}^{i}(\varepsilon) &= f^{i} + \varepsilon\\
\tilde{f}^{i+1}(\varepsilon) &= f^{i+1} - \varepsilon\frac{\theta^{i}-\theta^{i-1}}{\theta^{i+1}-\theta^{i-1}}.
\end{aligned}$$

(All other variables, e.g. the mechanism and support of F, are fixed at their optimal levels.) Note that the expected valuation is not affected by changes in ε and as θ^{i-1} and θ^{i+1} are *neighboring* elements of the true valuation support positive as well as negative ε are feasible (if not too large in absolute value).

Now consider the Lagrangian $\mathcal{L}$ of the maximization problem maximizing expected welfare over ε subject to (C) (fixing all other variables at their optimal level). From the definition $\tilde{f}^{i-1}$, $\tilde{f}^i$ and $\tilde{f}^{i+1}$, it is clear the $\mathcal{L}$ is linear in ε. As f^{i-1}, f^i and f^{i+1} are by assumption part of the optimal solution, $\mathcal{L}$ has to be maximized by $\varepsilon = 0$. As $\mathcal{L}$ is linear in ε and as ε in an open interval around 0 are feasible, this can only be the case if the derivative of $\mathcal{L}$ with respect to ε is zero everywhere. In the following it is shown that this is not possible.

Suppose the derivative of $\mathcal{L}$ with respect to ε is zero everywhere. For $\varepsilon = 0$, we have

[14] By proposition 2, there can be at most one signal between θ^{i-1} and θ^{i+1}.

$VV(\theta^{i-1},0) < VV(\theta^i,0) < VV(\theta^{i+1},0)$ by lemma 3 (where $VV(\theta^k,\varepsilon) = (1+\lambda)(\theta^k - c/I) - \lambda(\theta^{k+1}-\theta^k)(1-\tilde{F}(\theta^k,\varepsilon)/\tilde{f}^k(\varepsilon)$ denotes the virtual valuation of θ^k for a given ε). As ε increases the virtual valuations change as $\tilde{f}^{i-1}$ and $\tilde{f}^{i+1}$ decrease while f^i increases. Denote by $\varepsilon' > 0$ the lowest ε such that (at least) one of the following conditions is met

- $VV(\theta^i,\varepsilon) = VV(\theta^{i+1},\varepsilon)$
- $\tilde{f}^{i-1}(\varepsilon) = 0$.

For concreteness, let the first condition be met at ε', i.e. $VV(v_i,\varepsilon') = VV(v_{i+1},\varepsilon')$. Note that the value of $\mathcal{L}$ at $\varepsilon = \varepsilon'$ is the same as at $\varepsilon = 0$ as the derivative of $\mathcal{L}$ with respect to ε is supposed to be zero. As a next step (which will again not change the value of $\mathcal{L}$), change $y(\theta_j,\cdot)$ in case $\theta_j \in \{\theta^i,\theta^{i+1}\}$ to $\tilde{y}(\theta_j,\theta_{-j}) = \tilde{y}(\theta_{j+1},\theta_{-j}) = y(\theta^i,\theta_{-j})\tilde{f}^i/(\tilde{f}^i + \tilde{f}^{i+1}) + y(\theta^{i+1},\theta_{-i})\tilde{f}^{i+1}/(\tilde{f}^i + \tilde{f}^{i+1})$ (neglecting the argument "(ε')" of $\tilde{f}^{\cdot}$ for readability).[15] This change will not affect $\mathcal{L}$ as $\mathcal{L}$ is linear in $y(\theta_j,\theta_{-j})$ with slope equal to the virtual valuation and both θ^i and θ^{i+1} had the same virtual valuation. As a last step, note that – following the proof of lemma 3 – merging types θ^i and θ^{i+1} to $\theta^i\tilde{f}^i/(\tilde{f}^i+\tilde{f}^{i+1}) + \theta^{i+1}\tilde{f}^{i+1}/(\tilde{f}^i+\tilde{f}^{i+1})$ with probability $\tilde{f}^i(\varepsilon') + \tilde{f}^{i+1}(\varepsilon')$ will not affect expected welfare but relax the budget constraint, see the proof of lemma 3. Hence, the value of $\mathcal{L}$ increases due to this change. However, this contradicts that at the optimal solution $\mathcal{L}$ is maximized by the "optimal" values θ^{i-1}, θ^i, θ^{i+1} and f^{i-1}, f^i, f^{i+1} (holding all other variables at their optimal values).

If the other condition is met at ε', i.e. $\tilde{f}^{i-1}(\varepsilon') = 0$, the last step of the proof is similar. If $\tilde{f}^{i-1}(\varepsilon') = 0$, eliminating θ^{i-1} will strictly increase $\mathcal{L}$ as θ^i's incentive compatibility constraint is strictly relaxed. □

Note that a monotone partition in case H has binary support can lead to at most three signals of which two would be the elements in the support of H and the third would be a convex combination of these two. This is exactly the situation ruled out by lemma 6. Therefore, the information structure maximizing expected welfare consists of at most two signals. □

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[15] It is understood that this change is repeated iteratively in case several agents have types in $\{\theta^i,\theta^{i+1}\}$.

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