On the Equivalence of Nash and Evolutionary Equilibrium in Finite Populations

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Abstract

This paper provides sufficient and partial necessary conditions for the equivalence of Nash and evolutionary equilibrium in symmetric games played by finite populations. The focus is on symmetric equilibria in pure strategies. The conditions are based on properties of the payoff function that generalize the constant-sum property and the "smallness" property, the latter of which is known from models of perfect competition and non-atomic, anonymous, or large games. The conditions are illustrated on examples of Bertrand and Cournot oligopoly games.

Keywords: Nash equilibrium, Evolutionary stability, Finite populations

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1 Nash Equilibrium and Evolution

The concept of Nash equilibrium is a cornerstone of game theory. From a dynamic point of view, (a subset of) Nash equilibria can be justified by evolutionary processes, some of which appeal to an infinitely large population of potential players. A reduction of such a dynamic approach to a static equilibrium notion is conceptualized in evolutionarily stable strategy (ESS; Maynard Smith and Price, 1973). An ESS is always a Nash equilibrium (NE); the ESS concept thus gives an evolutionary underpinning of the Nash equilibrium concept.

Schaffer (1988) proposed a generalization of the ESS concept that incorporates the possibility of finite population of players. Schaffer and many others since then have shown that, in finite populations, the generalized ESS differs from Nash equilibrium for many games. This is because in finite populations relative payoff becomes important. Like Maynard Smith and Price's notion of an ESS, Schaffer's generalized ESS consists of (i) an equilibrium condition and (ii) a stability condition. The concept obtained when the stability condition is omitted and only the equilibrium condition is retained is called (symmetric) evolutionary equilibrium (EE; Schaffer, 1989). Notice that the notions of EE and the ESS in finite populations make sense only for symmetric games, since payoffs of different players, the incumbent(s) and the mutant, are compared.

Previous contributions that considered the EE usually focused on a particular game, typically finding NE and EE and comparing them. For instance, Vega-Redondo (1996, Section 2.7.2) established for the example of oligopolistic competition in quantities that the EE corresponds to the Walrasian equilibrium and will always differ from NE (which, by definition, is the Cournot equilibrium). Similarly, Hehenkamp et al. (2004) showed that EE and NE differ for rent-seeking contests. In EE, expenditures are always higher than in NE.

In this paper we generalize the previous examples of the difference (or equivalence) of EE and NE. To this end we provide sufficient and partial necessary conditions on the payoff function that help examine whether the sets of evolutionary and of Nash equilibria coincide for a given game. The sufficient conditions generalize the results published in this journal by Ania (2008). It is sufficient for the equivalence of EE and NE that a game exhibits weak versions of competitiveness (which generalizes constant-sum) or of weak payoff externality (which generalize "smallness") properties at some symmetric strategy profiles. The necessary conditions, which are also based on these two properties, are new.

We illustrate the economic relevance of our conditions by means of the examples of Bertrand competition with constant unit cost and of Cournot competition. By relating Nash equilibrium to the concept of evolutionary equilibrium, we provide further dynamic backing for the Nash equilibrium concept in those classes of games where NE and EE coincide.

2 Equivalence of Nash and Evolutionary Equilibrium

2.1 Definitions and notation

We investigate a model where a set I, with $|I| = n$, of individuals play a symmetric n-person game $\Gamma = (I, \{X_i\}_{i=1,\dots,n}, \{\pi_i\}_{i=1,\dots n})$, where X_i denotes the strategy set of player i, $X := \times_{i=1}^{n} X_i$ the set of joint strategies profiles, and $\pi_i : X \to \mathbb{R}$ denotes the payoff function of player i. Symmetry requires that the strategy sets of players coincide ($X_i = X_j$ for all i, j), and that the payoff functions of different players are related ($\pi_i(x) = \pi_j(x')$ for all x, $x' \in X$ such that x' is the (i,j)-permutation of the vector x). Let $\mathcal{G}$ denote the class of games under consideration.

We do not make any assumptions on X or on π_i. The set X can be finite or infinite, with or without further structure. Functions π_i can be continuous or not.

Let $[a]^k$ denote object a repeated k times, i.e. $[a]^k = a, \overset{k}{\ldots}, a$.

Definition 1 *Symmetric strategy profile $a = ([a]^n)$ is a* ***symmetric pure Nash equilibrium*** *of game Γ if*

$$\pi_1(a, [a]^{n-1}) \geq \pi_1(b, [a]^{n-1}) \text{ for all } b \in X_1.$$

The set of all symmetric pure Nash equilibria in game Γ is denoted by X^N.

Definition 2 *Symmetric strategy profile $a = ([a]^n)$ is an* ***evolutionary equilibrium*** *of game Γ if*

$$\pi_1([a]^{n-1}, b) \geq \pi_1(b, [a]^{n-1}) \text{ for all } b \in X_1.$$

The set of all evolutionary equilibria in game Γ is denoted by X^E.

Since in symmetric games $\pi_1([a]^{n-1}, b) = \pi_i(b, [a]^{n-1})$ for $i \neq 1$, the latter definition compares payoffs of different players after a deviation of one player from a to b. In the evolutionary interpretation, if a deviator to b has higher payoff than the players that stay at a, then a is not viable and thus cannot be evolutionary equilibrium.

Now we introduce the concepts of weak payoff externalities and weak competitiveness. Let $x \in X^n$ be an arbitrary strategy profile and let $x' = (x'_i, x_{-i})$ denote the profile in which $x'_i \neq x_i$, while $x'_j = x_j$ for $j \neq i$.

Definition 3 *Consider game $\Gamma \in \mathcal{G}$.*

(a) *Game* Γ *has* ***weak payoff externality between*** x ***and*** x'***, and between players*** i, j *if*

$$|\pi_i(x) - \pi_i(x_i', x_{-i})| > |\pi_j(x) - \pi_j(x_i', x_{-i})| \,. \qquad \text{(WPE)}$$

(b) *Game* Γ *has* ***weak payoff externalities*** *if (WPE) holds for all* x, x' *and for all pairs of players* (i, j), $j \neq i$.

Part (b) of the definition appears in Ania (2008). The definition means that when player i changes strategy, the effect on own payoff is larger than the effect on the payoff of any other player j. Weak payoff externality can be seen as a generalization of smallness in competitive markets. There a decision by one competitor does not affect other players at all. Here it may affect other players, but the effect on own payoff is larger than on others' payoff.

Definition 4 *Consider game* $\Gamma \in \mathcal{G}$.

(a) *Game* Γ *is* ***weakly competitive between*** x ***and*** x'***, and between players*** i ***and*** j *if*

$$\begin{array}{l} \pi_i(x) - \pi_i(x_i', x_{-i}) \geq 0 \Rightarrow \pi_j(x) - \pi_j(x_i', x_{-i}) \leq 0 \text{ and} \\ \pi_i(x) - \pi_i(x_i', x_{-i}) < 0 \Rightarrow \pi_j(x) - \pi_j(x_i', x_{-i}) \geq 0. \end{array} \qquad \text{(WC)}$$

(b) *Game* Γ *is* ***weakly competitive*** *if for all* x, x' *and all players* i*, there exists player* $j \neq i$ *such that (WC) holds.*

The definition extends the notion of a strictly competitive game for the general n-player case, $n \geq 2$.[1] If player i wins from a deviation, then at least one other player does not win. If player i loses from a deviation, at least one player does not lose. Observe that the class of weakly competitive games includes constant-sum games as a special case.

Part (b) of the two definitions each requires the respective property to hold globally in a game, i.e. for all strategy profiles. For our purposes it will be sufficient that the properties hold locally between symmetric profiles $x = ([a]^n)$ and some one-player deviations from them $x' = (b, [a]^{n-1})$.

[1] (See e.g. Friedman, 1990, Ch. 3).

2.2 Sufficient conditions for equilibrium equivalence

Theorem 1 shows that the local versions of both structural properties (weak payoff externalities and weak competitiveness) represent sufficient conditions for the equivalence of Nash and evolutionary equilibrium. Moreover, the theorem establishes that the two properties can be locally substituted for each other.

Theorem 1 *Suppose that in a game $\Gamma \in \mathcal{G}$, for each $x = ([a]^n)$ and for $x' = (b, [a]^{n-1})$ such that $\pi_1(b, [a]^{n-1}) > \pi_1([a]^n)$, either (WPE) or (WC) (or both) hold for players 1 and i, $i = 2, \ldots, n$. Then the sets of symmetric pure Nash equilibria and evolutionary equilibria coincide, i.e. $X^N = X^E$.*

Proof. Consider a monomorphic profile $x = ([a]^n)$. Suppose $x \in X^N$ but $x \notin X^E$. It follows that

$$\pi_1([a]^n) \geq \pi_1(b, [a]^{n-1}) \text{ for all } b$$

and that there exists $b \in X$, $b \neq a$ such that

$$\pi_1(b, [a]^{n-1}) > \pi_1([a]^{n-1}, b). \quad (1)$$

Hence

$$\pi_1([a]^n) - \pi_1(b, [a]^{n-1}) < \pi_1([a]^n) - \pi_1([a]^{n-1}, b).$$

By symmetry of the game $\pi_1([a]^n) = \pi_i([a]^n)$ and $\pi_1([a]^{n-1}, b) = \pi_i(b, [a]^{n-1})$, for $i \neq 1$, thus

$$\pi_1([a]^n) - \pi_1(b, [a]^{n-1}) < \pi_i([a]^n) - \pi_i(b, [a]^{n-1}).$$

The last inequality violates (WPE) (the left-hand side is non-negative, and thus so is the right-hand side). Then (WC) has to be satisfied for these a, b, i.e.

$$\pi_1([a]^n) \geq \pi_1(b, [a]^{n-1}) \Rightarrow \pi_i([a]^n) \leq \pi_i(b, [a]^{n-1}).$$

By symmetry of the game $\pi_i([a]^n) = \pi_1([a]^n)$ and $\pi_i(b, [a]^{n-1}) = \pi_1([a]^{n-1}, b)$. Then

$$\pi_1([a]^n) \leq \pi_1([a]^{n-1}, b)$$

and by (1),

$$\pi_1([a]^n) < \pi_1(b, [a]^{n-1}),$$

which violates the assumption that $x \in X^N$. Thus $x \in X^E$.

Suppose now that $x \in X^E$ but $x \notin X^N$. It follows that

$$\pi_1([a]^{n-1}, b) \geq \pi_1(b, [a]^{n-1}) \text{ for all } b$$

and that there exists $b \neq a$ such that

$$\pi_1(b, [a]^{n-1}) > \pi_1([a]^n). \tag{2}$$

Hence

$$\pi_1(b, [a]^{n-1}) - \pi_1([a]^n) \leq \pi_1([a]^{n-1}, b) - \pi_1([a]^n).$$

By symmetry of the game, for $i \neq 1$,

$$\pi_1(b, [a]^{n-1}) - \pi_1([a]^n) \leq \pi_i(b, [a]^{n-1}) - \pi_i([a]^n),$$

which contradicts (WPE). Then Γ must be weakly competitive for these a, b. Hence

$$\pi_1([a]^n) < \pi_1(b, [a]^{n-1}) \Rightarrow \pi_i([a]^n) \geq \pi_i(b, [a]^{n-1}).$$

Again by symmetry,

$$\pi_1([a]^n) \geq \pi_1([a]^{n-1}, b),$$

and by (2)

$$\pi_1(b, [a]^{n-1}) > \pi_1([a]^{n-1}, b),$$

which yields a contradiction to $x \in X^E$. Thus $x \in X^N$. ■

Corollary 1 *Let $\Gamma \in \mathcal{G}$ be weakly competitive. Then $X^N = X^E$.*

The theorem generalizes the results in Ania (2008, Propositions 1 and 2) in three ways. First, the constant-sum property is replaced by the much weaker property of weak competitiveness. Second, the properties of weak competitiveness and weak payoff externalities need to hold only at symmetric strategy profiles and profitable one-player deviations from them. Third, we show that the two properties can be locally substituted for each other. Accordingly, the theorem also covers games that possess locally a mixture of weak competitiveness and weak payoff externalities properties. The examples below and in Section 2.3 illustrate this.

Example 1 *Game with a pair of strategy profiles x, x' that neither satisfy (WPE) nor (WC) between x and x', but that satisfies the assumptions of Theorem 1.*

Consider the symmetric three-person game with payoff matrices (Player 3 chooses between matrices)

	3:α	
1\2	α	β
α	$1,1,1$	$3,0,3$
β	$0,3,3$	$6,6,9$

	3:β	
1\2	α	β
α	$3,3,0$	$9,6,6$
β	$6,9,6$	$5,5,5$

.

Consider profiles (α, α, β) with payoffs $(3, 3, 0)$ and (β, α, β) with payoffs $(6, 9, 6)$. The change in strategy of Player 1 increases own payoff by 3, and increases the payoff of the other two players by 6. Thus the two profiles satisfy neither (WPE) nor (WC). Furthermore, (α, α, α) and (β, α, α) violate (WPE) but satisfy (WC), and (β, β, β) and (α, β, β) violate (WC) but satisfy (WPE). The game is neither weakly competitive nor does it have weak payoff externalities, but by Theorem 1 we have $X^N = X^E (= \{(\alpha, \alpha, \alpha)\})$.

2.3 Bertrand competition

The importance of the requirement that only profitable one-player deviations from symmetric strategy profiles need satisfy either (WPE) or (WC) is illustrated referring to the standard economic example of Bertrand oligopoly with constant unit cost.

Example 2 *Bertrand oligopoly with constant unit cost.*

Consider a symmetric n-firm Bertrand oligopoly with market demand $Q(p)$ and cost function $C_i(q_i) = cq_i$, for all firms $i = 1, \ldots, n$. Firms set prices $p_i \geq 0$ and market demand is shared equally between all firms that charge the lowest price $p^{\min} = \min_i p_i$,

$$q_i(p_1, \ldots, p_n) = \begin{cases} \frac{Q(p^{\min})}{\#I(p_1,\ldots,p_n)} & \text{if } p_i = p^{\min} \\ 0 & \text{otherwise} \end{cases},$$

where $\#I(p_1, \ldots, p_n)$ denotes the number of firms charging the lowest price. Accordingly, firm i's profit function reads $\pi_i(p_1, \ldots, p_n) = q_i(p_1, \ldots, p_n)(p_i - c)$. Standard assumptions guarantee that there exists a unique symmetric Bertrand equilibrium, at which all firms price at marginal cost, i.e. $X^N = \{(c, \ldots, c)\}$.

From the equilibrium profile, any unilateral price reduction changes only the payoff of the deviating firm, thus (WPE) is satisfied (in fact, (WC) also trivially holds then). Any unilateral price increase is weakly competitive (but does not exhibit weak payoff externalities). From symmetric profiles with prices below marginal cost, a unilateral price increase changes the payoff of the deviating firm to zero, while the other firms have a larger loss. A price reduction will make the deviating firm have larger loss, while the other firms will increase profit to zero. Both deviations are therefore weakly competitive. From symmetric profiles with prices above marginal cost, any unilateral profitable price reduction makes the payoff of all other firms drop to zero. Thus such a deviation is weakly competitive.

Theorem 1 therefore implies:

Corollary 2 *The model of Bertrand oligopoly has* $X^E = X^N = \{(c, \ldots, c)\}$*.*

Note that a weaker version of Theorem 1 that would require one of the properties to hold for all one-player deviations from symmetric profiles would not be sufficient to claim the result because a non-profitable reduction involving a price above marginal cost satisfies neither (WPE) nor (WC).

2.4 Necessary conditions for equilibrium equivalence

The second theorem identifies the two above-mentioned structural properties to be locally necessary for some symmetric strategy profiles for the equivalence between Nash and evolutionary equilibrium, provided that the game is generic in a certain sense. The conditions apply to equilibrium profiles; for non-equilibrium profiles, weak payoff externalities and weak competitiveness still turn out to be partially necessary.

Theorem 2 *Let* $\Gamma \in \mathcal{G}$ *be such that* $b \neq a$ *implies* $\pi_1(b, [a]^{n-1}) \neq \pi_1([a]^n)$ *and* $\pi_1(b, [a]^{n-1}) \neq \pi_i(b, [a]^{n-1})$ *for all* $i \neq 1$*. Suppose that the sets of symmetric pure Nash equilibria and evolutionary equilibria coincide, i.e.* $X^N = X^E$ *in* Γ*. Then*

i) For each $a \in X$ *such that* $x = ([a]^n) \in X^N = X^E$*, for all* $x' = (b, [a]^{n-1})$*, either (WPE) or (WC) (or both) hold between* x *and* x'*.*

ii) For each $a \in X$ *such that* $x = ([a]^n) \notin X^N = X^E$*, we have*

$$\min_{b \in X} \pi_1(b, [a]^{n-1}) < \pi_1([a]^n) < \max_{b \in X}(b, [a]^{n-1})$$

or there exists $x' = (b, [a]^{n-1})$ *such that either (WPE) or (WC) (or both) hold between* x *and* x'*.*

Proof. Consider $x = ([a]^n) \in X^N = X^E$. Then $\pi_1([a]^n) \geq \pi_1(b, [a]^{n-1})$ and $\pi_1([a]^{n-1}, b) \geq \pi_1(b, [a]^{n-1})$ for any b. By symmetry and genericity, the second inequality is strict. Hence

$$\pi_1([a]^n) - \pi_1(b, [a]^{n-1}) > \pi_1([a]^n) - \pi_1([a]^{n-1}, b)$$

By symmetry, $\pi_1([a]^n) = \pi_i([a]^n)$ and $\pi_1([a]^{n-1}, b) = \pi_i(b, [a]^{n-1})$, $i \neq 1$, and thus

$$\pi_1([a]^n) - \pi_1(b, [a]^{n-1}) > \pi_i([a]^n) - \pi_i(b, [a]^{n-1}).$$

If $\pi_i([a]^n) - \pi_i(b, [a]^{n-1}) \geq 0$, then (WPE) is satisfied between x and $x' = (b, [a]^{n-1})$. If $\pi_i([a]^n) - \pi_i(b, [a]^{n-1}) < 0$, then (WC) is satisfied between x and x' (and possibly also (WPE) if $|\pi_1([a]^n) - \pi_1(b, [a]^{n-1})| > |\pi_i([a]^n) - \pi_i(b, [a]^{n-1})|$).

Consider $x = ([a]^n) \notin X^N = X^E$. Then there exists b such that $\pi_1(b, [a]^{n-1}) > \pi_1([a]^n)$. Therefore $\pi_1([a]^n) < \max_{b \in X} \pi_1(b, [a]^{n-1})$. The condition in ii) then claims that it is impossible that $\pi_1([a]^n) = \min_{b \in X} \pi_1(b, [a]^{n-1})$ and for all $x' = (b, [a]^{n-1})$ both (WPE) and (WC) are violated between x and x'. To obtain a contradiction, suppose that the above conditions hold. By genericity, $\pi_1([a]^n) < \pi_1(b, [a]^{n-1})$ for all $b \neq a$. Since (WC) is violated, we have $\pi_i([a]^n) - \pi_i(b, [a]^{n-1}) \leq 0$ for all $i \neq 1$, all $b \neq a$. Since (WPE) is violated, it hence follows that $\pi_1(b, [a]^{n-1}) - \pi_1([a]^n) \leq \pi_i(b, [a]^{n-1}) - \pi_i([a]^n)$ for all $i \neq 1$, all $b \neq a$. By symmetry, $\pi_i([a]^n) = \pi_1([a]^n)$ and $\pi_i(b, [a]^{n-1}) = \pi_1([a]^{n-1}, b)$. Thus

$$\pi_1(b, [a]^{n-1}) \leq \pi_1([a]^{n-1}, b) \text{ for all } b \neq a.$$

But this means that $x \in X^E$, a contradiction. ■

The theorem provides possible means to argue that a given game has evolutionary equilibria different from Nash equilibria. If Nash equilibria of the game are known, one method is to show that for a given Nash equilibrium profile $([a]^n)$ both (WPE) and (WC) are violated for some b. Without knowing the equilibria, one way to show that the game cannot have $X^N = X^E$ is to use a necessary condition for the equilibrium point and again show that both (WPE) and (WC) are violated for some b. Alternatively, using condition ii) of the theorem, one can find a symmetric profile $([a]^n)$ for which $\pi_1([a]^n) = \max_{y \in X}(y, [a]^{n-1})$ or $\pi_1([a]^n) = \min_{y \in X}(y, [a]^{n-1})$ and (WPE) and (WC) are violated for all b. The application of the theorem and the necessity of conditions are illustrated on the examples below.

Example 3 *Games with continuous strategy sets and payoff functions (e.g. Cournot oligopoly).*

Consider any game with continuous strategy sets $X_i \subset \mathbb{R}$ and differentiable payoff functions $\pi_i : X \to \mathbb{R}$. At an interior Nash equilibrium x^N it holds that $\frac{\partial \pi_i}{\partial x_i}(x^N) = 0$. Suppose that x_i^N is an isolated maximum of $\pi_i(x_i, x_{-i}^N)$ and that $\frac{\partial \pi_j}{\partial x_i}(x^N) \neq 0$ (i.e. (WPE) is violated at x^N). If $\frac{\partial \pi_j}{\partial x_i}(x^N) > 0$, then for x_i' smaller than but sufficiently close to x_i^N, we obtain $\pi_i(x_i', x_{-i}^N) < \pi_i(x^N)$ and $\pi_j(x_i', x_{-i}^N) < \pi_j(x^N)$, i.e. both (WPE) and (WC) are violated. If $\frac{\partial \pi_j}{\partial x_i}(x^N) < 0$, then for x_i' larger than but arbitrarily close to x_i^N both (WPE) and (WC) are violated. Thus in any such game $X^E \neq X^N$.

We thus obtain from Theorem 2:

Corollary 3 *Let $X_i \subset \mathbb{R}$ be continuous and $\pi_i : X \to \mathbb{R}$ differentiable for all players i. Further, let $x^N \in int\left(X^N\right)$ be an isolated interior Nash equilibrium. Then the following implication of equilibrium equivalence holds:*

$$X^E = X^N \quad \Longrightarrow \quad \frac{\partial \pi_j}{\partial x_i}(x^N) = 0.$$

In particular, consider a symmetric Cournot oligopoly with payoff function $\pi_i(q_1, \ldots, q_n) = P(Q) \cdot q_i - C(q_i)$. Suppose it has an interior Nash equilibrium q^N and that demand is strictly decreasing at the equilibrium point, $P'(Q^N) < 0$, where $Q^N = n \cdot q^N$. Since for $j \neq i$ and $q_i > 0$ we have $\frac{\partial \pi_j}{\partial q_i} = \frac{\partial \pi_i}{\partial q_j} = P'(Q) \cdot q_i$, it follows that $\frac{\partial \pi_j}{\partial q_i}(q^N) \neq 0$ and thus $X^E \neq X^N$.

Example 4 *Game with the necessary condition ii) violated.*

Consider the symmetric two-player game with payoff matrix

	α	β
α	$5,5$	$3,6$
β	$6,3$	$2,2$

.

For this game, $\pi_1(\beta, \beta) = \min_{y \in X} \pi_1(y, \beta)$. Furthermore, profile (β, β) satisfies neither (WPE) nor (WC) with respect to (α, β). Thus $X^E \neq X^N$ in this game. (In fact, $X^E = (\beta, \beta)$ and $X^N = \varnothing$.)

3 Conclusion

We have identified the classes of symmetric games where the sets of (symmetric) evolutionary and Nash equilibria in pure strategies coincide. The key properties are weak payoff externalities and weak competitiveness, both of which independently guarantee equilibrium equivalence (Theorem 1). Although these classes may appear narrow, they contain such important classes as constant-sum, strictly competitive, and non-atomic games. In these classes, Nash equilibrium has a more solid evolutionary (dynamic) background, as it is likely to be stable with respect to both introspective and imitative learning processes (see Bergin and Bernhardt, 2004).[2]

Ania (2008) identified the global properties of weak payoff externalities or constant-sum as being independently sufficient for equilibrium equivalence. Our results add that the properties of weak payoff externalities and weak competitiveness (which is a generalization of the constant-sum property) can be locally substituted for each other and that it is sufficient if they apply only to profitable deviations. Bertrand oligopoly with constant unit costs represents an economic example where the full force of the extensions is required to establish equilibrium equivalence.

The structural properties of weak payoff externalities and weak competitiveness are not only sufficient for equilibrium equivalence, but also necessary at equilibrium profiles. They are also partially necessary at other profiles in the sense of Theorem 2. Our necessary conditions provide clear predictions for games with continuous strategy space and differentiable payoff functions. Although the conditions may be not always easy to check, the results in this paper can be helpful in the dynamic analyses of games.

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