# Cournot Competition under Uncertainty*

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**Abstract**

We analyze Cournot competition under demand uncertainty. We show that under rather general assumptions, the game has no asymmetric equilibria but multiple symmetric equilibria. Multiplicity is caused by the requirement of nonnegative prices and remains an issue also for simple demand specifications, such as the linear case. We then show that uniqueness of equilibrium is guaranteed if uncertainty is resolved after production has taken place but prior to the sales decision, which is often referred to as the free disposal case. Production is higher under free disposal than in any equilibrium of the game without free disposal.

**Keywords:** Demand uncertainty, Cournot competition,
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# 1 Introduction

In oligopolistic markets, firms often face considerable uncertainty upon production. This may concern the exact demand realization, their competitors' costs or even components of the own production cost. A large literature has recognized these issues and has addressed questions as information acquisition, information sharing, or strategic experimentation.[1]

This paper focuses on an aspect that has widely been ignored by the literature on imperfect competition under demand uncertainty, the constraint that prices cannot become negative. We show that under very general conditions Cournot competition under demand uncertainty has multiple symmetric equilibria but no asymmetric equilibria and we provide an intuitive characterization of equilibrium production in the general case. We then show that in a rather general framework uniqueness of Cournot equilibrium is guaranteed also for uncertain demand if firms may dispose of produced quantity after uncertainty unraveled.

There is a small literature dealing with nonnegativity constraints in models with linear demand and cost functions that complement our analysis. Malueg et al. (1998a,b) demonstrate by numerical examples that standard results on information sharing may break down if one accounts for the nonnegativity constraint on prices. Lagerlöf (2006a) shows that multiple equilibria may be an issue even in simple settings with linear demand and cost functions, demonstrating that the origin of multiple equilibria is the nonnegativity constraint rather than general demand and cost functions. Lagerlöf (2006b) provides conditions on the distribution of uncertainty such that in a linear framework uniqueness can be guaranteed.

Our model contributes to the literature in two important ways. First, we provide a tractable framework for the analysis of quantity competition under demand uncertainty and characterize all (of the possibly multiple) equilibria of the game. Second, we provide a plausible setting in which uniqueness of the Cournot equilibrium under demand uncertainty is guaranteed independently of the distribution of uncertainty.

The paper is organized as follows. In section 2 we state the model. In section 3 we analyze the Cournot market game under demand uncertainty and characterize the equilibria of the game. In section 4 we analyze the case of free disposal and establish uniqueness of equilibrium in this case. Section 5 concludes.

# 2 The Model

We consider a market game where $n$ symmetric firms simultaneously produce a homogenous good. Denote by $q = (q_1, \dots, q_n)$ the vector of outputs of the $n$ firms, and let $Q = \sum_{i=1}^{n} q_i$

[1] For example Ponssard (1979), Gal-Or (1985, 1986), Vives (1984, 1990), and many others.

be total quantity produced in the market. We assume that all firms have the same cost function which we denote by $C(q_i)$. Inverse Demand is given by the function $P(Q,\theta)$, which depends on total quantity $Q \in \mathbb{R}^+$, and the random variable $\theta \in \mathbb{R}$ which represents uncertainty. The random variable $\theta \in \mathbb{R}$ is distributed according to a distribution $F(\theta)$ with bounded support.[2] We introduce the parameter $z \leq 0$ as a lower bound on market prices in order to take into account nonnegativity of prices ($z = 0$) or disposal cost ($z < 0$) and denote the quantity where this lower bound is met by $\bar{Q}(\theta)$.[3]

Assumption 1 *$P(Q,\theta)$ is differentiable in $\theta$ with $P_\theta(Q,\theta) > 0$ for all $\theta \in \mathbb{R}$. At each realization of $\theta$, the following regularity assumptions have to be satisfied for quantities $Q < \bar{Q}(\theta)$:*

*(i) $P(Q,\theta)$ is continuously differentiable[4] in $Q$ with[5] $P_q(Q,\theta) < 0$.*

*(ii) $C(q_i)$ is twice continuously differentiable in $q_i$ and nondecreasing.*

*(iii) $\lim_{Q\to\infty} P(Q,\theta) < z$ for all $\theta \in \mathbb{R}$.*

In our model, firms decide on production before the realization of $\theta$ is known. Firm $i$'s expected profit from operating if production is $q$ is given by

$$\pi_i(q) = \int_{-\infty}^{\infty} P(Q,\theta)\, q_i(\theta)\, dF(\theta) - C(q_i). \tag{1}$$

Throughout the paper we consider only those cases where production is *gainful*, i. e. $\mathbf{E}[P(0,\theta)] > C_q(0)$. We are interested in pure strategy Nash equilibria of the game. We denote an equilibrium by $q^*$ and the corresponding total equilibrium output by $Q^*$.

# 3 Equilibrium Analysis

Our first theorem establishes existence of a symmetric equilibrium, characterizes equilibrium production in the Cournot market game under demand uncertainty and shows that no asymmetric equilibria exist.

[2]While $F$ has bounded support, it will be convenient to assume that $P(Q,\theta)$ is defined for all $\theta \in \mathbb{R}$ and $Q \in \mathbb{R}_+$.

[3]In case the lower bound is not binding we can set $\bar{Q}(\theta) = \infty$.

[4]Differentiability is not crucial for our results but makes exposition easier.

[5]Throughout the paper we denote the derivative of a function $g(x,y)$ with respect to an argument $m$, $m = x, y$, by $g_m(x,y)$, the second derivative with respect to that argument by $g_{mm}(x,y)$, and the cross derivative by $g_{xy}(x,y)$.

Theorem 1 *The Cournot market game with uncertain demand has no asymmetric equilibria and at least one symmetric equilibrium. In any equilibrium expected marginal revenue equals marginal cost, i.e. total production is characterized as follows,*[6]

$$Q^* = \left\{ Q : \int_{\tilde{\theta}(Q)}^{\infty} \left[ P(Q,\theta) + P_q(Q,\theta)\frac{Q}{n} \right] dF(\theta) = C_q\left(\frac{Q}{n}\right) \right\}, \tag{2}$$

*where $\tilde{\theta}$ is the demand scenario from which on production is binding.*[7]

Proof See appendix A. □

Obviously, in all cases with positive production, the first order condition equates expected marginal revenue of production with marginal cost. Note that theorem 1 covers degenerate uncertainty as a special case. Then, equation (2) is the standard first order condition in Cournot equilibrium.

Notice that we cannot establish uniqueness of equilibrium. Let us emphasize that it is in the first place the demand uncertainty in connection with the nonnegativity constraint on prices that gives rise to multiplicity of equilibria (rather than our weak assumptions on demand in a particular demand scenario). This has been pointed out by Lagerlöf (2006a). He shows by example that also under much more restrictive assumptions the Cournot oligopoly game with uncertain demand may still have multiple equilibria. In particular, he gives an example for multiple equilibria in case of *linear demand* and a two–state distribution and conjectures in a companion paper (Lagerlöf (2006b), p. 34): [This distribution] "could be approximated with a continuous–state two–hump distribution that satisfies all the differentiability and full support assumptions made here, and which would therefore also give rise to multiple equilibria." He then shows that in a model with linear demand a sufficient condition for uniqueness is that the distribution of $\theta$ is such that its hazard rate is either (i) monotone or (ii) its slope is changing its sign exactly once and is first negative and then positive (Lagerlöf (2006b), Proposition 1). Our paper is complementary to Lagerlöf's contribution in the sense that we provide a characterization of all equilibria of the game in the presence of possibly multiple equilibria. In the next section we show that under quite plausible conditions Cournot competition under demand uncertainty has a unique equilibrium for rather general demand specifications and any distribution of uncertainty.

[6] In the following we set $z = 0$, which means that prices are nonnegative. This is the most natural case. In the appendix, a characterization is provided for general values of $z$.

[7] That is, $\tilde{\theta} = \{\theta : P(Q,\theta) = 0\}$. Note that here we have set $z = 0$ for easier exposition. The proof in the appendix deals with the more general case $z \leq 0$. For $z$ low enough it would always hold that $\tilde{\theta} = \underline{\theta}$.

# 4 Free Disposal

We now consider the case that uncertainty is resolved after the firms' production decisions, but before firms decide on the quantities they want to sell. We assume that disposal is costless.[8] Firms may benefit from strategic withholding in particular if demand turns out to be low, since they have the opportunity to raise the price above zero, i.e. make positive profits.

In the presence of free disposal we have to analyze a two stage game. At the first stage firms decide on production quantities $q = (q_i)_{i=1,\ldots,n}$. Then, they learn the true state of the world, $\theta$. Once they know the state of the world, they decide on the quantities $y(q,\theta) = (y_i(q,\theta))_{i=1,\ldots,n}$ they want to sell. We make the following additional regularity assumptions that have to be satisfied at each $\theta \in \mathbb{R}$ for quantities $Q < \bar{Q}(\theta)$:

Assumption 2 *(i) $P(Q,\theta)$ satisfies $P_q(Q,\theta) + P_{qq}(Q,\theta)q_i < 0$.*

*(ii) $P(Q,\theta)q_i$ is (differentiable) strict supermodular in $q_i$ and $\theta$, i. e. $\frac{d^2[P(Q,\theta)q_i]}{dq_i d\theta} > 0$ for all $i$, $\theta$, and $q_{-i}$.*[9,10]

*(iii) $C(q_i)$ is convex.*

We get the following result:

Theorem 2 *(i) The Cournot market game with free disposal has a unique equilibrium $q^{*D}$ which is symmetric. Total equilibrium production is uniquely characterized by*

$$Q^{*D} = \left\{ Q : \int_{\theta^n(q)}^{\infty} \left[ P(Q,\theta) + P_q(Q,\theta)\frac{Q}{n} \right] dF(\theta) = C_q\left(\frac{Q}{n}\right) \right\}, \tag{3}$$

*where $\theta^n(q)$ is the demand scenario from which on all firms sell their entire production.*

*(ii) For any finite $n$, equilibrium production is higher in the case with free disposal than in the case without free disposal, i.e. $Q^{*D} > Q^*$.*

Proof See appendix B. □

Note that the first order condition in theorem 2 follows the same intuition as in the case without free disposal. However, unlike in theorem 1, we can establish uniqueness

[8] We do so mainly for easier exposition. On may also assume that firms incur different cost of selling a unit or disposal of a unit. Uniqueness is guaranteed whenever selling cost is higher than disposal cost.

[9] Throughout the paper $q_{-i}$ denotes the quantities produced by the firms other than $i$, and $Q_{-i} = \sum_{j \neq i} q_j$.

[10] Part (ii) of the assumption is not essential. It makes it, however, much easier to write down expected profits.

of equilibrium. Moreover, if firms can freely dispose of units they do not want to sell, equilibrium production is always higher than in the case without free disposal. The intuition is that high production does not imply zero prices (and thus, profits) in the case of low demand since firms can always raise the price above zero by withholding some of their production from the market.

## 5 Conclusion

In this paper we have characterized all equilibria of a Cournot market game under demand uncertainty. We have shown that the game has a unique equilibrium if firms may freely dispose of produced quantity, while it has multiple symmetric (but no asymmetric) equilibria if all produced quantity is necessarily offered for sale in any demand scenario. Production is higher in the case of free disposal, since too high production (relative to the demand scenario) is not as harmful for the firm in terms of profits.

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# A Proof of Theorem 1

Denote by $\tilde{\theta}(q) = \{\max \theta : P(Q, \theta) = z\}$ the demand scenario where the price rises above $z$ (we then say that production is "binding"). The profit function is given by[11]

$$\pi_i(q) = \int_{-\infty}^{\tilde{\theta}(Q)} z q_i dF(\theta) + \int_{\tilde{\theta}(Q)}^{\infty} P(Q, \theta) q_i dF(\theta) - C(q_i). \tag{4}$$

**Existence** In order to prove existence we apply theorem 2.1 of Amir and Lambson (2000), p. 239. They show that the standard Cournot oligopoly game has at least one symmetric equilibrium and no asymmetric equilibria whenever demand $P(\cdot)$ is continuously differentiable and decreasing, cost $C(\cdot)$ is twice continuously differentiable and nondecreasing and, moreover, the cross partial derivative $\frac{d\pi_i(q)}{dQ_{-i}dQ} > 0$, where $Q$ denotes total production and $Q_{-i}$ production of the firms other than $i$. In order to see that the conditions required by Amir and Lambson are satisfied in our setup, note that in our game firms choose output given *expected* marginal cost and *expected* demand. Expected inverse demand is given by

$$EP(Q) = \int_{-\infty}^{\tilde{\theta}(Q)} z dF(\theta) + \int_{\tilde{\theta}(Q)}^{\infty} P(Q, \theta) \, dF(\theta). \tag{5}$$

Note that $\frac{dEP(Q)}{dQ} = \int_{\tilde{\theta}}^{\infty} P_q(Q, \theta) dF(\theta) < 0$, and thus, $EP(Q)$ is strictly decreasing in $Q$. Moreover, the cross partial derivative[12]

$$\frac{d\pi^2(q)}{dQ_{-i}dQ} = - \int_{\tilde{\theta}(Q)}^{\infty} P_q(Q, \theta) dF(\theta) > 0$$

is positive. Thus, by Amir and Lambson (2000), theorem 2.1, there exists at least one symmetric equilibrium and no asymmetric equilibria.

**Characterization** Since no asymmetric equilibria exist in our framework, we now focus on the symmetric case and characterize equilibrium production. The derivative of $\pi_i(q)$ (equation (4)) with respect to $q_i$ is given by

$$\frac{d\pi_i}{dq_i} = \int_{-\infty}^{\tilde{\theta}(Q)} z dF(\theta) + \int_{\tilde{\theta}(Q)}^{\infty} \left[ P_q(Q, \theta) q_i + P(Q, \theta) \right] dF(\theta) - C_q\left(\frac{Q}{n}\right), \tag{6}$$

Note that $\frac{d\pi_i}{dq_i} > 0$ at $Q = 0$ (since production is assumed to be gainful), that $\frac{d\pi_i}{dq_i} < 0$ for some finite value of $Q$, and that $\frac{d\pi_i}{dq_i}$ is continuous. Thus, we have at least one point

[11] Note that for $\theta < \theta^Q(q)$ profits are zero if $z = 0$.

[12] See Amir and Lambson (2000), p. 238.

where (2) is satisfied and $\frac{d\pi_i}{dq_i}$ is decreasing. The characterization of equilibrium as given in theorem 1 follows straightforwardly (for easier exposition we have set $z$ equal to zero in the theorem).

# B Proof of Theorem 2

## B.1 Proof of Part (i)

### The Second Stage of the Game — the Sales Decision

In the first step we characterize the sales decision at stage two (that is constrained by the firms' production decisions at stage one) for each $\theta$. Note that in principle production choices at stage one may be asymmetric. In order to simplify the exposition we will order the firms according to their production levels, i. e. $q_1 \leq q_2 \leq \cdots \leq q_n$, throughout the proof. In order to demonstrate that our results also hold if the firms incurs sales cost at stage two, we introduce sales cost $S(q_i)$, which are assumed to be convex and identical for all firms.

An equilibrium of the second stage in scenario $\theta$ given production choices $q$, $y^*(q, \theta)$, satisfies simultaneously for all firms

$$y_i^*(q,\theta) \in \arg\max_{\mathrm{y}} \left\{ P(\mathrm{y} + y_{-i}^*, \theta))\mathrm{y} - S(\mathrm{y}) \right\} \qquad \text{s.t.} \quad 0 \leq \mathrm{y} \leq q_i. \tag{7}$$

Note that at very low values of $\theta$ (i.e. in case of very low demand) it will be profitable to withhold produced quantity from the market. Sale choices in this case are given by the equilibrium quantities of a Cournot game with certain demand $P(Q, \theta)$ and symmetric marginal cost $S_q(y_i)$. By assumption 2 the equilibrium of this game [which we denote by $\tilde{y}^{*0}(\theta)$] is unique and symmetric for each $\theta \in [-\infty, \infty]$.[13] From (7) it follows that $\tilde{y}_i^{*0}(\theta)$ is implicitly determined by the first order condition

$$P(n\tilde{y}_i^{*0}, \theta) + P_y(n\tilde{y}_i^{*0}, \theta)\tilde{y}_i^{*0} = S_y(\tilde{y}_i^{*0}).$$

Now as $\theta$ increases, at some critical value that we denote by $\theta^1(q)$, firm 1 (the one with the lowest production) becomes constrained (i.e. the firm's equilibrium quantity in the above Cournot game is above its production). The critical demand scenario is implicitly determined by $q_1 = y_1^{*0}(\theta^1)$. If it holds that $q_1 < q_2$, then at $\theta^1(q)$ only firm one becomes constrained. Then, in equilibrium, firm 1 sells its whole production whereas the remaining firms sell less than their production. Their sales are determined by their equilibrium output

[13] See, for example Selten (1970), or Vives (2001), pp. 97/98.

of a Cournot game among $n-1$ firms given the residual demand $P(Q-q_1,\theta)$ [denoted by $\tilde{y}_i^{*1}(q,\theta)$], which solves the first order condition

$$P(q_1+(n-1)\tilde{y}_i^{*1},\theta)+P_y(q_1+(n-1)\tilde{y}_i^{*1},\theta)\tilde{y}_i^{*1}=S_y(\tilde{y}_i^{*1}).$$

The sales equilibrium in the case where one firm is constrained by its production is a vector $y^{*1}(q,\theta)$, where $y_i^{*1}(q,\theta)=\min\{q_i,\tilde{y}^{*1}(q,\theta)\}$.

As $\theta$ increases further, we pass through $n+1$ cases, from case "0" (no firm sell its entire production) to case "$n$" (all $n$ firms sell their entire production). Note that two critical values $\theta^m(q)$ and $\theta^{m+1}(q)$ coincide whenever $q_m=q_{m+1}$, and that it holds that $\theta^m(q)<\theta^{m+1}(q)$ (by assumption 1) whenever $q_m<q_{m+1}$.

Now we are prepared to characterize the equilibrium sales decisions in case "$m$" where $m$ firms sell their entire production. In this case, the $m$ firms with the lowest production quantities sell their entire production, whereas the $n-m$ remaining firms sell

$$\begin{aligned}\tilde{y}_i^{*m}(q,\theta) &= \left\{y_i\in\mathbb{R}:P\left(\sum_{i=1}^{m}q_i+(n-m)\,y_i,\theta\right)\right. \\ &\qquad \left.+P_y\left(\sum_{i=1}^{m}q_i+(n-m)\,y_i,\theta\right)y_i=S_y\left(y_i\right)\right\},\end{aligned}\tag{8}$$

The equilibrium sales quantities in case $m$ firms sell their entire production are given by

$$y_i^{*m}(q,\theta)=\min\{q_i,\tilde{y}_i^{*m}(q,\theta)\},\tag{9}$$

and aggregate sales in this case are

$$Y^{*m}(x,\theta)=\sum_{i=1}^{n}y_i^{*m}(q,\theta).\tag{10}$$

This allows us finally to pin down the profit of firm $i$ in scenario "$m$",

$$\pi_i^{*m}(q,\theta)=\begin{cases}P\left(Y^{*m},\theta\right)q_i-S\left(q_i\right) & \text{if } i\leq m,\\ \\ P\left(Y^{*m},\theta\right)\tilde{y}_i^{*m}(q,\theta)-S\left(\tilde{y}_i^{*m}(q,\theta)\right) & \text{if } i>m.\end{cases}\tag{11}$$

Note that it holds that $\frac{d\pi_i^{*m}}{dq_i}>0$ only if $i\leq m$, and $\frac{d\pi_i^{*m}}{dq_i}=0$ otherwise, since higher production affects a firm's sales decision at stage two only in case the firm already sold its entire production. Obviously, in this case the derivative must be positive.

The following two properties, which we need in order to analyze stage one of the game, follow from the above analysis of stage two:

Property 1 (Monotonicity of $\theta^m$) *$\frac{d\theta^m(q)}{dq_i}$ is strictly positive if $i\leq m$ (i.e. if firm $i$ sells its entire production), and zero otherwise.*

Proof $\theta^m(q)$ is the demand realization from which on firm $m$ wants to sell its entire production. At $\theta^m(q)$ it holds that $y_i^*(\theta^m(q)) = \tilde{y}_i^{*m}(\theta^m(q)) = q_m$ for all $i \geq m$ and $y_i^*(\theta^m(q)) = q_i < q_m$ for all $i < m$. Thus, $\theta^m(q)$ is implicitly defined by the conditions

$$P\left(\sum_{i=1}^{m} q_i + (n-m)q_m, \theta^m(q)\right) + P_q\left(\sum_{i=1}^{m} q_i + (n-m)q_m, \theta^m(q)\right) q_m - S_q(q_m) = 0.$$

Differentiation with respect to $q_i$, $i < m$, yields

$$P_q(\cdot) + P_\theta(\cdot)\frac{d\theta^m(q)}{dq_i} + P_{qq}(\cdot)\, q_m + P_{q\theta}(\cdot)\, q_m \frac{d\theta^m(q)}{dq_i} = 0,$$

and solving for $\frac{d\theta^m(q)}{dq_i}$ we obtain

$$\frac{d\theta^m(q)}{dq_i} = -\frac{P_q(\cdot) + P_{qq}(\cdot)\, q_m}{P_\theta(\cdot) + P_{q\theta}(\cdot)\, q_m} > 0$$

due to assumption 2.

Differentiation with respect to $q_i$, $i = m$, yields

$$(n-m+2)P_q(\cdot) + P_\theta(\cdot)\frac{d\theta^m(q)}{dq_i} + (n-m+1)P_{qq}(\cdot)\, q_m + P_{q\theta}(\cdot)\, q_m \frac{d\theta^m(q)}{dq_i} - S_{qq}(\cdot) = 0,$$

and solving for $\frac{d\theta^m(q)}{dq_i}$ we obtain

$$\frac{d\theta^m(q)}{dq_i} = -\frac{(n-m+2)P_q(\cdot) + (n-m+1)P_{qq}(\cdot)\, q_m - S_{qq}(\cdot)}{P_\theta(\cdot) + P_{q\theta}(\cdot)\, q_m} > 0,$$

also due to assumption 2. Finally, differentiation with respect to $q_i$, $i > m$, yields

$$P_\theta(\cdot)\frac{d\theta^m(q)}{dq_i} + P_{q\theta}(\cdot)\, q_m \frac{d\theta^m(q)}{dq_i} = 0,$$

which implies that $\frac{d\theta^m(q)}{dq_i} = 0$ for $i > m$. □

Property 2 [Properties of Marginal Profits at Stage Two] *Suppose all firms but firm 1 have produced similar quantities, which is summarized in the vector $q_{-1}^0$. Firm 1 has produced $q_1$, less than each of the other firms. We obtain:*

*(i)* $\frac{d\pi_1^{*1}(q_1^0, q_{-1}^0, \theta)}{dq_1} \geq 0$ *for* $\theta^1 \leq \theta \leq \theta^n$.

*(ii)* $\frac{d\pi_1^{*n}(q_1', q_{-1}^0, \theta)}{dq_1} \geq \frac{d\pi_1^{*n}(q_1'', q_{-1}^0, \theta)}{dq_1} \geq 0$ *for* $q_1' < q_1''$, $\theta^n \leq \theta \leq \infty$.

Proof (i) The first part holds due to the fact in case firm 1 is sells its entire production, i. e. $(\theta \geq \theta^1)$, firm 1 would like to sell more than $q_1$ for all demand realizations $\theta \geq \theta^1$, which, however, is not possible due to its low production quantity.

(ii) The first inequality follows from concavity of the profit functions at the second stage, which is implied by assumption 2. Thus, the first order condition at stage two for each $\theta$ is decreasing in $q_1$ until $\tilde{y}_i^{*0}$, which immediately yields the first inequality of part (ii). The second inequality is due to the fact that in case all firms are selling their entire production, i. e. ($\theta \in [\theta^n, \infty]$), firm 1 would like to sell more for all demand realizations $\theta$ (which is not possible because it is constrained by its low production). □

**The First Stage - Production Decision**

We now analyze the first stage of the game, i.e. the firms' production decisions. The results obtained for the sales stage enable us to derive a firm $i$'s profit from producing quantity $q_i$, given that the other firms produce $q_{-i}$ and sales choices at stage two are given by $y^{*m}(q, \theta)$ for $\theta \in [\theta^m(q), \theta^{m+1}(q)]$. Recall that upon production the firms face demand uncertainty. Thus, a firm's profit from given levels of production $q$ is the integral over equilibrium profits at each $\theta$ given $q$ on the domain $[-\infty, \infty]$, taking into account the probability distribution over the demand scenarios. For each $\theta$, firms anticipate equilibrium sales quantities at the second stage as characterized in the previous section. Note that for any $q > 0$ all firms sell less than their production if $\theta$ is sufficiently low. As $\theta$ increases, more and more firms sell their entire production. Thus, a tuple of production levels that initially gave rise to an equilibrium where no firm is constrained, then leads to an equilibrium where first one, then two, three, ..., and finally $n$ firms sell their entire production. In order to simplify the exposition we define $\theta^0 \equiv -\infty$ and $\theta^{n+1} \equiv \infty$. Then, the profit of firm $i$ is given by[14]

$$\pi_i(q, y^*) = \sum_{m=0}^{m=n} \int_{\theta^m}^{\theta^{m+1}} \pi_i^{*m}(q, \theta) dF(\theta) - C(q_i). \tag{12}$$

Note that at each critical value $\theta^m$, $m = 1, \ldots, n$ it holds that $\pi^{*m-1}(q, \theta^m) = \pi^{*m}(q, \theta^m)$. Thus, $\pi_i(q, y^*)$ is continuous. Differentiating $\pi_i(q, y^*)$ yields[15]

$$\frac{d\pi_i(q, y^*)}{dq_i} \sum_{m=i}^{n} \int_{\theta^m(q)}^{\theta^{m+1}(q)} \frac{d\pi_i^{*m}(q, \theta)}{dq_i} dF(\theta) - C_q(q_i) \tag{13}$$

We prove part (i) of the theorem in two steps. In part I we show existence and in part II uniqueness of equilibrium.

[14]Note that it is never optimal for a firm to be unconstrained at $\infty$ and thus, we always obtain $\theta^n \leq \infty$.

[15]Note that continuity of $\pi_i$ implies that due to Leibnitz' rule the derivatives of the integration limits cancel out. Moreover, $\pi_i^{*m}$ only changes in $q_i$ if firm $i$ is constrained in scenario $m$, i. e. $i \leq m$. Thus, the sum does not include the cases where firm $i$ is unconstrained, i. e. $m < i$.

**Part I: Existence of Equilibrium** In the following we show that a symmetric equilibrium of the two stage Cournot market game exists, and that equilibrium choices $q_i^* = \frac{1}{n}Q^*$, $i = 1, \ldots, n$, are implicitly defined by equation (3). For this purpose it is sufficient to show quasiconcavity of firm $i$'s profit given the other firms invest $q_{-i}^*$, $\pi_i(q_i, q_{-i}^*)$.

Note that $\pi_i(q_i, q_{-i}^*)$ is defined piecewisely. For $q_i < q_i^*$, we have to examine to profit of firm 1 (by convention the firm with the lowest production) given that $q_2 = q_3 = \cdots = q_n$. This implies that $\theta^2 = \cdots = \theta^n$ and it follows from (12) that

$$\begin{aligned}\pi_1(q_1, q_{-1}^*) &= \int_{-\infty}^{\theta^1(q)} \pi_1^{*0}(q,\theta)dF(\theta) + \int_{\theta^1(q)}^{\theta^n(q)} \pi_1^{*1}(q,\theta)dF(\theta) \\ &\quad + \int_{\theta^n(q)}^{\infty} \pi_i^{*n}(q,\theta)dF(\theta) - C(q_1).\end{aligned} \tag{14}$$

For $q_i > q_i^*$, the profit of firm $i$ is the profit of the firm with the highest production (firm $n$ according to our convention), given all other firm have produced the same, i. e. $q_1 = \cdots = q_{n-1}$. We get

$$\begin{aligned}\pi_n(q_n, q_{-n}^*) &= \int_{-\infty}^{\theta^{n-1}(q)} \pi_n^{*0}(q,\theta)dF(\theta) + \int_{\theta^{n-1}(q)}^{\theta^n(q)} \pi_n^{*n-1}(q,\theta)dF(\theta) \\ &\quad + \int_{\theta^n(q)}^{\infty} \pi_n^{*n}(q,\theta)dF(\theta) - C(q_1).\end{aligned} \tag{15}$$

**(i) The shape of $\pi_i(q_i, q_{-i}^*)$ for $q_i > q_i^*$:** The second derivative of the profit function $\pi_n$ is given by

$$\frac{d^2\pi_n}{(dq_n)^2} = -\frac{d\theta^n(q)}{dq_n} \underbrace{\left[\frac{d\pi_n^{*n}(q,\theta^n)}{dq_n}\right]}_{=0\ (q_n \text{ is opt. at} \theta^n)} f(\theta^n) + \int_{\theta^n(q)}^{\infty} \underbrace{\frac{d^2\pi_n^{*n}(q,\theta)}{(dq_n)^2}}_{<0 \text{ by A1 part (iv)}} f(\theta)d\theta < 0. \tag{16}$$

Note that the first term cancels out and the second term is negative by concavity of the stage two profit function (implied by assumption 2). We find that for $q_i \geq q_i^*$, $\pi_i(q_i, q_{-i}^*)$ is concave, which implies that upwards deviations are not profitable.

**(ii) The shape of $\pi_i(q_i, q_{-i}^*)$ for $q_i < q_i^*$:** This region is more difficult to analyze since the profit function $\pi_1(q_1, q_{-1}^*)$ is not concave. We can, however, show quasiconcavity of $\pi_1(q_1, q_{-1}^*)$. For this purpose we need property 2 below in order to complete the proof of existence (part I). We can show quasiconcavity of $\pi_1(q_1, q_{-1}^*)$ by showing that

$$\frac{d\pi_1(q_1^0, q_{-1}^*)}{dq_1} > \frac{d\pi_1(q_1^*, q_{-1}^*)}{dq_1} = 0 \qquad \text{for all} \quad q_1^0 < q_1^*.$$

This holds true, since [compare also equation (13)]

$$
\begin{aligned}
\frac{d\pi_1(q_1^0,q_{-1}^*)}{dq_1} &= \underbrace{\int_{\theta^1(q_1^0,q_{-1}^*)}^{\theta^n(q_1^0,q_{-1}^*)} \frac{d\pi_1^{*1}(q_1^0,q_{-1}^*,\theta)}{dq_1} dF(\theta)}_{\geq 0 \quad \text{by property 2, part (i)}} + \int_{\theta^n(q_1^0,q_{-1}^*)}^{\infty} \frac{d\pi_1^{*n}(q_1^0,q_{-1}^*,\theta)}{dq_1} dF(\theta) \\
&\geq \int_{\theta^n(q_1^0,q_{-1}^*)}^{\infty} \frac{d\pi_1^{*n}(q_1^0,q_{-1}^*,\theta)}{dq_1} dF(\theta) \\
&= \underbrace{\int_{\theta^n(q_1^0,q_{-1}^*)}^{\theta^n(q_{-1}^*,q_{-1}^*)} \frac{d\pi_1^{*n}(q_1^0,q_{-1}^*,\theta)}{dq_1} dF(\theta)}_{\geq 0 \quad \text{by properties 1 and 2, part (ii)}} \\
&\quad + \underbrace{\int_{\theta^n(q_1^*,q_{-1}^*)}^{\infty} \left[\frac{d\pi_1^{*n}(q_1^0,q_{-1}^*,\theta)}{dq_1} - \frac{d\pi_1^{*n}(q_1^*,q_{-1}^*,\theta)}{dq_1}\right] dF(\theta)}_{>0 \quad \text{by property 2, part (ii)}} \\
&\quad + \underbrace{\int_{\theta^n(q_1^*,q_{-1}^*)}^{\infty} \frac{d\pi_1^{*n}(q_1^*,q_{-1}^*,\theta)}{dq_1} dF(\theta)}_{=\frac{d\pi_i(q^*)}{dq_i}=0 \text{ [recall that } \theta^1(q^*)=\theta^n(q^*)]} \geq 0.
\end{aligned}
$$

To summarize, in part I (i) and (ii) we have shown that $\pi_i(q_i, q_i^*)$ is quasiconcave. We conclude that the first order condition

$$
\begin{aligned}
\frac{d\pi_i\left(q, y^*\right)}{dq_i} &= \int_{\theta^n(q)}^{\infty} \frac{d\pi_i^{*n}\left(q,\theta\right)}{dq_i} dF\left(\theta\right) - C_q\left(q_i\right) \\
&= \int_{\theta^n(q)}^{\infty} \left[P(Q,\theta) + P_q(Q,\theta)q_i - S(q_i)\right] dF\left(\theta\right) - C_q\left(q_i\right) = 0
\end{aligned}
\tag{17}
$$

as given in the theorem indeed characterizes equilibrium production in the Cournot market game. Note that for easier exposition we do not account for sales cost in the theorem.

**Part II: Uniqueness** In this part we show that (i) $q^*$ is the unique symmetric equilibrium and (ii) that there are no asymmetric equilibria.

**(i) $q^*$ is the unique symmetric equilibrium.** If production quantities are equal, i. e. $q_1^0 = q_2^0 = \cdots = q_n^0$, we have

$$
\frac{d\pi_i(q^0)}{dq_i} = \int_{\theta^n(q^0)}^{\infty} [P(nq_i^0,\theta) + P_q(nq_i^0,\theta)q_i^0 - S_q(q_i^0)] f(\theta)d\theta - C_q(q_i^0).
$$

Differentiation yields[16]

$$
\frac{d^2\pi_i(q^0)}{(dq_i)^2} = \int_{\theta^n(q^0)}^{\infty} \left[(n+1)P_q(nq_i^0,\theta) + nP_{qq}(nq_i^0,\theta)q_i^0 - S_{qq}(q_i^0)\right] dF(\theta) - C_{qq}(q_i^0) < 0,
$$

[16]Differentiation works as in (16).

which is negative due to assumption 2. Thus, since $\frac{d\pi_i(q^*)}{dq_i} = 0$ and moreover $\pi_i(q)$ is concave along the symmetry line, no other symmetric equilibrium can exist.

**(ii) There cannot exist an asymmetric equilibrium.** Any candidate for an asymmetric equilibrium $\hat{q}$ can be ordered such that $\hat{q}_1 \leq \hat{q}_2 \leq \cdots \leq \hat{q}_n$, where at least one inequality has to hold strictly. This implies $\hat{q}_1 < \hat{q}_n$. The profit of firm $n$ can be obtained by setting $i = n$ in equation (12), and the first derivative is given by

$$\frac{d\pi_n}{dq_n} = \int_{\theta^n(q)}^{\infty} \frac{d\pi_n^{*n}(q,\theta)}{dq_n} f(\theta)d\theta - C_q(q_n).$$

It is easy to show that firm $n$'s profit function is concave by examination of the second derivative [see equation (16)]. Thus, any asymmetric equilibrium $\hat{q}$, if it exists, must satisfy $\frac{d\pi_n(\hat{q})}{dq_n} = 0$. We now show that whenever it holds that $\frac{d\pi_n(\hat{q})}{dq_n} = 0$, firm 1's profit is increasing in $q_1$ at $\hat{q}$ (which implies that no asymmetric equilibria exist).

From equation (13) it follows that the first derivative of firm 1's profit function is given by

$$\frac{d\pi_1}{dq_1} = \int_{\theta^1(q)}^{\theta^2(q)} \frac{d\pi_1^{*n}(q,\theta)}{dq_1} f(\theta)d\theta + \cdots + \int_{\theta^n(q)}^{\infty} \frac{d\pi_1^{*n}(q,\theta)}{dq_1} f(\theta)d\theta - C_q(q_1).$$

Note that all the integrals in $\frac{d\pi_1}{dq_1}$ are positive since firm 1 is constrained at all demand realizations and therefore would want to increase its sales. Thus, we have

$$\frac{d\pi_1}{dq_1} > \int_{\theta^n(q)}^{\infty} \frac{d\pi_1^{*n}(q,\theta)}{dq_1} f(\theta)d\theta - C_q(q_1),$$

where the RHS are simply the last two terms of $\frac{d\pi_1}{dq_1}$. Note furthermore that $\hat{q}_1 < \hat{q}_n$ also implies that $C_q(\hat{q}_1) < C_q(\hat{q}_n)$ (due to assumption 2 part (iii)) and

$$\frac{d\pi_1(\hat{q})}{dq_1} = P(\hat{q},\theta) + P_q(\hat{q},\theta)\hat{q}_1 - S_q(\hat{q}_1) > P(\hat{q},\theta) + P_q(\hat{q},\theta)\hat{q}_n - S_q(\hat{q}_n) = \frac{d\pi_n(\hat{q})}{dq_n}$$

Now we can conclude that

$$\frac{d\pi_1}{dq_1} > \int_{\theta^n(q)}^{\infty} \frac{d\pi_1^{*n}(q,\theta)}{dq_1} f(\theta)d\theta - C_q(q_1) > \int_{\theta^n(q)}^{\infty} \frac{d\pi_n^{*n}(q,\theta)}{dq_n} f(\theta)d\theta - C_q(q_n) = 0.$$

The equality is due to the fact that this part is equivalent to the first order condition of firm $n$, which is satisfied at $\hat{q}$ by construction. To summarize, we have shown that $\frac{d\pi_1}{dq_1} > 0$, which implies that there exist no asymmetric equilibria, since at any equilibrium candidate, firm 1 has an incentive to increase its production.

## B.2 Proof of Part (ii)

Consider the first order conditions that implicitly define total production in the cases without and with free disposal as given in theorems 1 and 2, part (i). Note that $\theta^n(q) > \tilde{\theta}(q)$ for all $q$ (since firms are constrained already in a lower demand scenario if they do not have the possibility to withhold quantity from the market once they have produced it). Furthermore, at (below, above) the demand realization $\theta^n(q^{*D})$ we have that $P_q(Q^{*D},\theta)\frac{Q^{*D}}{n} + P(Q^{*D},\theta) - S_q(\frac{1}{n}Q^{*D}) = 0$ $(< 0, > 0)$. Thus, the lefthand–sides of the first order conditions can be ordered as follows:

$$\begin{aligned} &\text{with free disposal:} && \int_{\theta^n(q)}^{\infty} \left[P_q(Q,\theta)\frac{1}{n}Q + P(Q,\theta) - S_q\left(\frac{1}{n}Q\right)\right] dF(\theta) \\ &\text{without free disposal:} && \geq \int_{\tilde{\theta}(Q)}^{\infty} \left[P_q(Q,\theta)\frac{1}{n}Q + P(Q,\theta) - S_q\left(\frac{1}{n}Q\right)\right] dF(\theta). \end{aligned} \tag{18}$$

Note that according to theorems 1 and 2, total production is determined as the values of $Q$ where the respective term equals $C_q\left(\frac{1}{n}Q^*\right)$ and $C_q\left(\frac{1}{n}Q^{*D}\right)$, respectively. Recall that in all cases we get interior solutions. Note that in the free disposal case, the LHS of the first order condition is decreasing in $Q$, while $C_q$ is increasing in $Q$, whereas in in the case without free disposal (second line of (18)) it satisfies $LHS(0) > C_q(0)$ (since production is gainful) and $LHS(Q) < C_q(Q)$ for $Q$ high enough. Since $C_q(Q)$ is increasing in $Q$, this immediately implies that for any equilibrium production $Q^*$ it holds that $Q^{*D} \geq Q^*$. Note that as $n$ approaches infinity, both first order conditions collapse to $\int_{-\infty}^{\infty}[P(Q,\theta) - S_q(0)]dF(\theta) = C_q(0)$.