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HUMAN CAPITAL, POLARIZATION, AND PARETO-IMPROVING ACTIVATING WELFARE

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Human Capital, Polarization, and Pareto-Improving Activating Welfare

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Abstract

Human capital not only generates market income but is a direct source of utility as well. The interaction between the non-economic motive for effort and the standard economic motive can generate multiple stationary solutions for individual household optimization. Depending on the initial distribution of skills, this multiplicity divides each group of otherwise identical households into two perpetually separated groups: one rich and educated, one poor and uneducated. If the rich have an interest in the education of the poor, polarized equilibria are typically Pareto-inefficient. While unconditional transfers only reduce the incentive of the uneducated to accumulate skills, there exist activating tax-transfer systems that Pareto-dominate any non-redistributing system. Transfers are transitory and there is a negative marginal income tax on household income below a certain threshold.

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1 Introduction

Most people value their own human capital beyond its role in generating income. While standard economic models recognize the role of human capital as an asset yielding a return by its capacity to increase wages, they generally do not focus on its role as a direct source of utility. In contrast, the present paper treats human capital as a direct source of utility. An individual exerts costly effort to accumulate human capital not only to generate higher future income and consumption (the economic motive) but also because of the direct utility effect of increased human capital (the non-economic motive). The contribution of this paper is twofold: First, to show that the interaction of these motives naturally leads to the existence of multiple stationary solutions to the individual optimization problem. Second, to provide a rational for tax systems with negative marginal taxes on low incomes.

Besides introducing human capital into the felicity function, this study retains standard assumptions of a canonical intertemporal household problem: Human capital generates income, whereas effort raises the stock of human capital and reduces felicity. The interpretation of human capital is inspired by psychology literature concerning human needs and subjective well-being. I argue that the conclusions of this literature support the following general assumptions: 1. The felicity function is additively separable with respect to consumption and human capital 2. The marginal utility of consumption decreases with increased consumption (Inada-conditions satisfied) 3. The marginal utility of human capital is neither particularly high at a low human capital nor does it decline as human capital rises (Inada-conditions violated). Within the simplest class of optimization problems that satisfy these requirements, multiple stationary solutions exist whenever the marginal utility of consumption is sufficiently small compared to the marginal disutility of effort at low consumption, while the marginal utility of human capital remains sufficiently large at higher levels. There is a threshold skill level: Households endowed with initial human capital below this level choose a path of increasing passivity and sustained poverty. Households endowed with initial human capital above the threshold, motivated both by economic and non-economic rewards, choose a path that leads to sustained activity, high skills, and income.

The multiplicity of stationary optima is a potential source of persistent inequality among inherently identical agents (e.g., equal except for initial skill). This result is concurrent with the large amount of literature on credit constraints and inequality, where inequality persists due to capital market imperfections. According to this literature, unskilled households remain unskilled because they cannot finance today's education using loans they would pay back their future increased income. The present paper does not account for physical capital or credit markets. However, self-financing constraints are not the reason for low-skill traps or segregation (the set of stationary optimum is robust with respect to the introduction of a perfect credit market). The reason is rather that skill cannot be bought, whether it be by current income, a money transfer, savings, or a loan. In case of multiple solutions to the household problem, low-skilled households do not raise their human capital because it is too difficult rather than too expensive. By reducing the marginal utility of consumption, an unconditional transfer undermines poor households' motivation to educate themselves. It turns any active low-skill household into an inactive household. In contrast, if it were credit constraints that caused the low-skill trap, the household would use a transfer to increase its human capital.

When simple unconditional transfers accentuate skill segregation, it seems natural to condition transfers upon the economic activity of the household. Indeed, many countries add economic activity incentives to their tax-transfer systems.[1] Surprisingly few theoretical models provide a rational for such schemes.[2] Once one recognizes the possibility of path dependent individually optimal effort and human capital accumulation, there emerges a wealth of arguments supporting activating welfare schemes and, in particular, for negative marginal income tax rates on low incomes. A particularly strong case for such schemes arises if one adds to the individual household problem described above a common interest in the education of the poor, shared also by the rich. Conditional redistribution then tends to Pareto-dominate non-redistributional tax systems. In the present paper such a common interest is generated by the presence of a public good: The larger the number of financially strong households that can participate in financing the public good, the better it is for each individual contributor. The rich are only willing to finance transitory transfers if this guarantees the education of the poor sufficiently quickly. If the willingness to pay is strong enough to compensate for the poor's disutility of the effort necessary to raise their skill, then polarization is Pareto-inefficient. The present paper shows that (in the presence of public goods) their exist tax-transfer schemes involving a negative marginal income tax for the initially unskilled that Pareto-improve any persistently polarized equilibrium.

The remainder of this paper is organized as follows. Section 2 introduces the individual optimiza-

[1]In particular, many countries have recently adopted negative marginal tax rates into their tax-transfer systems: the United States (earned income tax credit), the United Kingdom (working tax credit), Canada (working income tax benefit), Germany (combined wages), Ireland, New Zealand, Austria, Belgium, Denmark, Finland, France or the Netherlands. Some (notably Scandinavian) countries already have a tradition of activating welfare or "workfare".

[2]In the framework of Mirrlees' seminal article (Mirrlees [1971]) and most subsequent literature on optimal taxation, marginal income tax rates are always non-negative. An exception is Diamond [1980], who shows that if, instead of considering the intensive margin of labor choices, one considers the extensive margin (an individual worker faces the binary choice of working or not), negative marginal income tax rates can be optimal (see also Saez [2002], Choné and Laroque [2005], Laroque [2005]). Beaudry et al. [2009] deviate from the Mirrlees framework by assuming that the government is uninformed both about households' value of time in economic and non-economic activities and by allowing the government to condition not only on total incomes but also on individual wage rates. Beaudry et al. show that an optimal tax-transfer includes a negative marginal tax rate for workers with a wage below a certain cutoff rate.

tion problem Section 3 determined the conditions leading to multiplicity of stationary optima. Section 4 studies the effect of simple transfers, while Section 5 shows that, in the presence of public goods, conditional transfers can Pareto-improve any persistently polarized allocation. In the long-run the Pareto-improving allocation constructed (and implemented) in Section 5 generates equal skill (and felicity) for intrinsically equal agents (i.e. for agents that only differ with respect to initial skill. Section 6 first discusses the basic assumptions leading to the multiplicity of stationary solutions to the individual optimization problem at the core of the present paper (Section 6.1). Second, it shows that this multiplicity is robust with respect to the removal of credit market imperfections (Section 6.2). Third, section 6.3 discusses the possibility of addressing the issue of activating welfare from a (generalized) utilitarian perspective. Section 7 concludes and argues that the path dependence of individual skill accumulation can explain the popularity of conditional transfers even in the absence of public goods.

2 The individual household problem

The individual household solves

$$\max_{\{x_t\}_{t=0}} \int_0^\infty e^{-\rho t} u(c_t, x_t, h_t, G_t) dt \tag{1}$$

$$\begin{aligned} \text{subject to } \widehat{h}_t \quad &: \quad = \frac{dh_t/dt}{h_t} = g(x_t, h_t) = \gamma x_t - \delta h_t \text{ for all } t \geq 0, \\ c_t \quad &= \quad w h_t \text{ for all } t \geq 0 \\ & \quad\quad \text{given } h_0 \text{ and } \{G_t\}_{t=0}^\infty. \end{aligned}$$

where c_t (consumption) and x_t (effort) are control variables and h_t is a state variable (human capital). G_t is the amount of the (non-rival) public good consumed, which in the laissez-faire economy is zero or exogenous to the individual household. Household income wh_t depends on the household's skill and the general productivity level w. In the absence of credit markets the household simply consumes his current income ($c_t = wh_t$).

The instantaneous utility function is specified as

$$u(c_t, x_t, h_t, G_t) = \underbrace{\kappa \ln c}_{m(c)} + \underbrace{bh}_{f(h)} - \underbrace{\left[ax + (\alpha/2)x^2\right]}_{v(x)} + \xi G_t \text{ with } \kappa, b, a, \alpha > 0, \xi \geq 0, \tag{2}$$

where κ measures the strength of the economic motive for exerting effort, while b measures the intensity of the non-economic motive. a and α measure the cost of effort, where $\alpha > 0$ ensures the concavity of $v(x)$, which is necessary for the existence of a solution to (1). Note that if $\{G_t\}_{t=0}^\infty$ is exogenous for the individual household, it does not affect the solution of (1), even if $\xi > 0$.

The skill accumulation function is

$$g(x_t, h_t) = \gamma x_t - \delta h_t \text{ with } \gamma, \delta > 0, \tag{3}$$

where γ measures the effectiveness of the efforts to increase skills, *given* the current level of skill. δ measures depreciation of human capital. I assume that α is sufficiently large given ρ, a, κ, γ.[3]

The very fact that h enters the utility function, as well as the specific way it does so, may need particular justification. The former because it is unconventional, the latter because it is crucial for the results. Section 6.1 therefore provides some motivation for this paper's interpretation of h and the specific functional forms (2) and (3).

3 History-dependent optimal individual behavior

The section first solves the individual optimization problem (1) for an exogenously given path $\{G_t\}_{t=0}^{\infty}$, which does not affect optimal individual behavior. Lemma 1 first provides the common structure of the solution for all parameter combinations. The interplay of the standard terms $m(c)$ and $v(x)$ with the non-standard term $f(h) = b \cdot h$ in the optimization problem (1) gives rise to three cases, that are separately addressed in three propositions 2-4 below.

Lemma 1 *The first order conditions of (1) together with the accumulation rule (3) define the dynamic system*

$$\begin{cases} \dot{x} = \frac{v_x(\rho+\delta h)-(m_h+b)\gamma h}{\alpha} = \frac{(a+\alpha x)(\rho+\delta h)-(\kappa+bh)\gamma}{\alpha} \\ \dot{h} = (\gamma x - \delta h)h \end{cases} \tag{4}$$

The two isoclines are

$$\begin{cases} \dot{x} = 0 \text{ if } x_{\dot{x}=0}(h) := \frac{\gamma}{\alpha}\frac{\kappa+bh}{\rho+\delta h} - \frac{a}{\alpha} \\ \dot{h} = 0 \text{ if } h = 0 \text{ or } x_{\dot{h}=0}(h) := \frac{\delta}{\gamma}h \end{cases} \tag{5}$$

Depending on parameter values the system has no, one or two strictly positive stationary solutions. In addition there always is a trivial stationary state at $h_* = 0$. In all cases the maximization problem (1) has a unique solution $\{x(h_t), h_t\}_{t\geq 0}$ for any initial h_0, where $x(h)$ is a continuous policy function.

Proof:

The first order conditions of (1). The current value Hamiltonian of (1) is $H(x, h, G, \lambda) = m(wh) + bh - v(x) + \xi(G) + \lambda \cdot (\gamma x - \delta h)h$. Existence of an inner solution to $\max_x H(x, h, G, \lambda)$ requires $H_x = -v_x + \lambda\gamma h = 0$, which is sufficient since $H_{xx} = \alpha < 0$. At the maximum $v_x = \lambda\gamma h$. Taking the derivative with respect to time yields $v_{xx}\dot{x} = \dot{\lambda}\gamma h + \lambda\gamma\dot{h} = \dot{\lambda}\gamma h + v_x(\gamma x - \delta h)$. Inserting the adjoint equation $\dot{\lambda} = \rho\lambda - H_h = \rho\lambda - m_h - b - \lambda \cdot (\gamma x - 2\delta h)$ into this expression for $v_{xx}\dot{x}$ and using $v_x = \lambda\gamma h$ yields $v_{xx}\dot{x} = v_x(\rho + \delta h) - (m_h + b)\gamma h$. Together with the accumulation rule for human capital this

[3] Formally, $\alpha > (\rho a - \kappa\gamma)\, 8\gamma/\rho^2$ will ensure that the unstable stationary solution in the threshold case is a node.

defines the dynamic system

$$\begin{cases} \dot{x} = \frac{v_x(\rho+\delta h)-(m_h+b)\gamma h}{v_{xx}} \\ \dot{h} = (\gamma x - \delta h)h \end{cases}, \tag{6}$$

With $v(x) = ax + \frac{\alpha}{2}x^2$ two isoclines are

$$\begin{cases} \dot{x} = 0 \text{ if } x_{\dot{x}=0}(h) := \frac{(m_h+b)\gamma h}{\alpha(\rho+\delta h)} - \frac{a}{\alpha} \\ \dot{h} = 0 \text{ if } h = 0 \text{ or } x_{\dot{h}=0}(h) := \frac{\delta}{\gamma}h \end{cases} \tag{7}$$

Inserting $m(c) = \kappa \ln c$ yields (5).

Stationary solutions. The two isoclines always have an intersection at $h_* = 0$ defining the trivial stationary state of (4). The other stationary states are determined by the solutions to $\frac{\gamma}{\alpha}\frac{\kappa+bh}{\rho+\delta h} - \frac{a}{\alpha} = \frac{\delta}{\gamma}h$ or, rearranging, to

$$\frac{\delta^2}{\gamma}h^2 - \left(\frac{b\gamma - a\delta}{\alpha} - \frac{\delta}{\gamma}\rho\right)h - \frac{\kappa\gamma - \rho a}{\alpha} = 0.$$

With $C := \frac{\kappa\gamma-\rho a}{\alpha}$, $B := \left(\frac{b\gamma-a\delta}{\alpha} - \frac{\delta}{\gamma}\rho\right)$, and $D = \frac{\delta^2}{\gamma} > 0$ the (non-zero) stationary states of (4) are the solutions to

$$E(h) := C + Bh - Dh^2 = 0 \tag{8}$$

Equation (8) may have no, one or two strictly positive solutions

$$h_{1,2} = \frac{B \pm \sqrt{B^2 + 4DC}}{2D}.$$

Case 1 One strictly positive solution $h^* = \frac{B+\sqrt{B^2+4DC}}{2D} > 0$ if $C > 0$ or equivalently $\kappa > a\frac{\rho}{\gamma}$. In this case $x_{\dot{x}=0}(0) = \frac{\gamma}{\alpha}\frac{\kappa}{\rho} - \frac{a}{\alpha} > x_{\dot{h}=0}(0) = 0$ as in Figures 1a and 1b, such that the two isoclines have exactly one intersection at a strictly positive h, say $h^* > 0$.

Case 2 Two strictly positive solutions $h^{\text{th}} = \frac{B-\sqrt{B^2+4DC}}{2D} > 0$ and $h^* = \frac{B+\sqrt{B^2+4DC}}{2D} > h^{\text{th}}$ if $-\frac{B^2}{4D} < C < 0$ and $B > 0$. In this case $x_{\dot{x}=0}(0) < x_{\dot{h}=0}(0)$ and the $\dot{x} = 0$-isocline is rising sufficiently fast compared with the $\dot{h} = 0$-isocline as in Figure 2, such that there are two strictly positive intersections, $h^{\text{th}} > 0$ and $h^* > h^{\text{th}}$.

Case 3 No strictly positive solution. If $x_{\dot{x}=0}(0) < x_{\dot{h}=0}(0)$ and the $\dot{x} = 0$-isocline is not rising fast with h as in Figure 3, then there are no positive intersections.

Case 3a No real solution if $C < -\frac{B^2}{4D} < 0$.

Case 3b Two negative solutions if $-\frac{B^2}{4D} < C < 0$ and $B < 0$.

Stability of the solutions and existence of a continuous policy function. Appendix 8.1 shows that h_1 is unstable (whenever it exists, thus in Case 2) and h_2 is saddle-point stable (whenever it exists, thus in Cases 1 and 2). The trivial stationary state (h_*, x_*) is unstable in Case 1 and is a saddle point stable in Cases 2 and 3. Furthermore it is shown that any path satisfying (4) and converging to one of the three possible stationary solutions (including these stationary solutions) satisfy the transversality condition. Hartl et al. [2004] show that the there exists a continuos policy function through all stationary states, in all three cases, if in Case 2 with three stationary solutions $(h_* = 0 < h_1 < h_2)$, the unstable stationary solution lies in the concave domain of the Hamiltonian. Appendix 8.1 shows that this condition is met if α is sufficiently large (as assumed in footnote 3). ■

Intuition for equations (4) and (5).

Most of the ideas and results in this article can be understood by studying (4) and its graphical representations in Figures 1-3. For the $\dot{h} = 0$-isocline this is straightforward. It directly results from the assumed rule of skill accumulation $\dot{h} = (\gamma x - \delta h)h$ and consists of the vertical x-axis $(h = 0)$ in Figures 1-6 and the increasing straight line $x_{\dot{h}=0}(h) = \frac{\delta}{\gamma}h$. The larger the level of human capital, the larger the required effort to keep human capital at this level. As the arrows in Figures 1-3 indicate $\dot{h}_t > 0$ $(\dot{h}_t < 0)$ if (x_t, h_t) lies below (above) the $\dot{h} = 0$-isocline. The $\dot{h} = 0$-isocline is unchanged throughout the paper.

To gain an intuition for the behavior of $\dot{x}_t$ determined in (4), remember the familiar Euler-equations of the Ramsey-model: Starting with $c_t = c_{t+\Delta}$ for small $\Delta > 0$, current c_t should be reduced in favor of future $c_{t+\Delta}$ or equivalently $\frac{c_{t+\Delta}-c_t}{\Delta} \simeq \dot{c}_t > 0$ iff the benefit from delaying consumption (the gross interest rate) is larger than the cost of delaying (depreciation and felicity discounting). Analogously, in the present setting, starting with $x_t = x_{t+\Delta}$, current comfort should be reduced (x_t *raised*) in favor of future comfort ($x_{t+\Delta}$ reduced) or equivalently $\frac{x_{t+\Delta}-x_t}{\Delta} \simeq \dot{x}_t < 0$ iff the benefit from delaying comfort exceeds the cost of delaying. The benefit from exerting effort today rather than tomorrow is the felicity $(m_h + b)\Delta$ derived from each unit of additional skill between t and $t + \Delta$ times the number γh of such additional units produced by exerting one unit of additional effort today, thus $(m_h + b)\Delta \cdot \gamma h = \gamma(\kappa + bh)\Delta$. The cost of exerting effort today rather than tomorrow consist in skill depreciation and felicitiy discounting multiplied by the additional cost of effort $v_x \cdot (\rho + \delta h)\Delta$.[4] Thus $\dot{x}_t \lesseqqgtr 0$ iff $v_x(\rho + \delta h) \lesseqqgtr (m_h + b)\gamma h$. Since $v_x = a + \alpha x$ is increasing in x it follows that $\dot{x}_t > 0$ $(\dot{x}_t < 0)$ if (x_t, h_t) lies above (below) the above $\dot{x} = 0$-isocline as indicated by the corresponding arrows in the phase-diagrams (see for instance Figure 1).

[4] More precisely, the cost of additional effort today consists in direct cost $v_x\rho \cdot \Delta$ of exerting effort at t rather than at $t + \Delta$ plus the cost of reduced skill at $t + \Delta$ by depreciation. One unit of additional effort either at t or at $t + \Delta$ produces γh units of additional skill. If they are produced at t rather than at $t + \Delta$, the skill at $t + \Delta$ is smaller then when already produced at t by $\delta\gamma h^2\Delta$. This must be evaluated at the shadow price $\lambda_t \simeq \lambda_{t+\Delta}$, which is $\lambda_t = \frac{v_x}{\gamma h}$ (see Appendix 8.1). Thus the cost of exerting effort today due to depreciation is $\lambda_t \cdot \delta\gamma h^2 \cdot \Delta = v_x\delta h \cdot \Delta$.

The exact position of the $\dot{x} = 0$-isocline depends on the parameters (which change from case to case) and on the economic environment (which change from section to section). Raising the economic motive, by raising the marginal utility $m_h = \kappa/h$ for each h, raises the benefit from exerting effort today rather than tomorrow. Using the above intuition, this shifts upward the $\dot{x} = 0$-isocline. This will play a crucial role throughout the paper. In particular public policy will reduce (by transfers) or raise (by negative marginal taxes) the incentive to exert effort through its impact on m_h. Similarly an increase of γ (more efficient learning) or a reduction in a (lower disutility of effort) shift upward the $\dot{x} = 0$-isocline for any $h \geq 0$. In contrast, raising b (stronger non-economic motive) raises the incentive to exert effort today only for strictly positive h and does so the stronger, the larger h. An increase in b therefore raises the slope of the $\dot{x} = 0$-isocline without affecting its intercept $x_{\dot{x}=0}(h = 0)$. This reflects the Maslow hierarchy (see Section 6): The relative importance of the non-economic motive is negligible for small wh and rises with wh.

Case 1 arises if the economic motive is very strong (large $m_h = \kappa/h$ given γ, ρ, a, h). The household is active at any level of skill, irrespective of the strength of the non-economic motive (b). Case 3 arises when both motives for effort are weak (small κ and b). These two cases describe the range of possible solutions to the standard household problem in the absence of any non-economic motive ($b = 0$). The interesting threshold Case 2 occurs when the economic motive is not very strong (small κ), while the non-economic motive is large (large b). A low-skill household remains low skilled (because the non-economic reward is relatively small even with large b). A highly skill household remains highly skilled (because of the strong non-economic motive).

Case 1, Strong economic motive, high efficiency of learning, low discounting: Activation.

Proposition 2 *If the economic motive is sufficiently strong (Case 1), then the optimization problem (1) has a unique non-trivial stationary solution $h^* > 0$, which is a global attractor. Formally: If $\kappa > a\rho/\gamma$, then $\lim_{t\to\infty} h_t = h^* > 0$ for all $h_0 > 0$. See Figure 1.*

$\kappa > a\frac{\rho}{\gamma}$ whenever the economic motive for effort is strong (κ large), the household is patient (ρ small), it is easy to raise h (γ large) or the simple cost of effort is low (a small). As a short-cut, I will say in this case that the "economic motive for effort is strong".

Proof. Corollary of Lemma 1. If $\kappa > a\frac{\rho}{\gamma}$ (Case 1), then $x_{\dot{x}=0}(0) = \frac{\gamma}{\alpha}\frac{\kappa}{\rho} - \frac{a}{\alpha} > x_{\dot{h}=0}(0) = 0$ as in Figures 1a and 1b then the two isoclines have exactly one intersection at a strictly positive h, say $h^* > 0$. Note that for this case, at least qualitatively, it is irrelevant whether $x_{\dot{x}=0}(h)$ is increasing (as in Figure 1a) or decreasing (as in Figure 1b). ∎

Case 2, Weak economic motive, strong non-economic motive: threshold dynamics

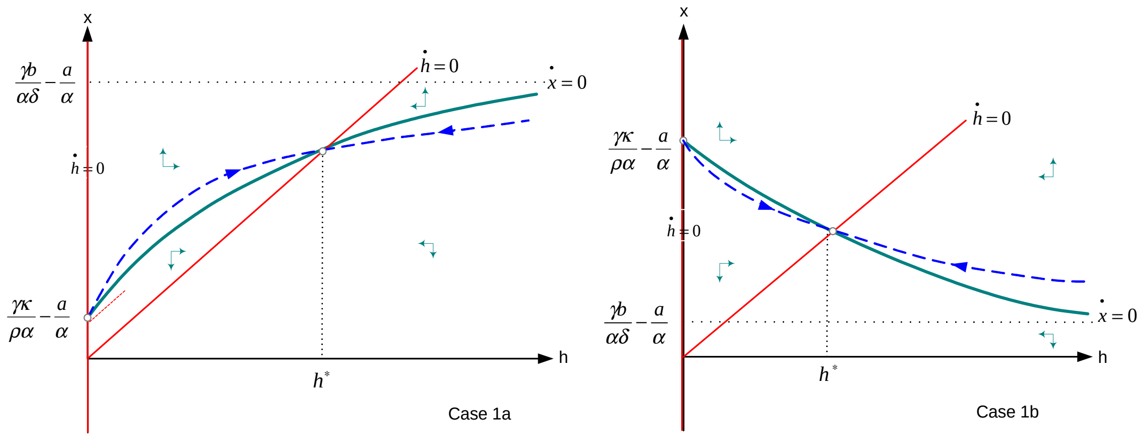

Figure 1: Case 1

Proposition 3 *If the economic motive is not dominant and the non-economic motive is sufficiently strong (Case 2), then the optimal policy has a threshold above which human capital converges to a large stationary state and below which human capital converges to a low stationary state. Formally: If* $\kappa < \min\{a\rho/\gamma, b\rho/\delta\}$ *and* $b > \frac{2\delta\sqrt{\alpha\gamma(a\rho-\gamma\kappa)}+\alpha\rho\delta+a\delta\gamma}{\gamma^2}$*, then* $\lim_{t\to\infty} h_t = h_* = 0$ *for all* $h_0 < h^{th}$ *and* $\lim_{t\to\infty} h_t = h^*$ *for all* $h_0 > h^{th}$*, where* $x(h)$ *is a continuous policy function. See Figure 2.*

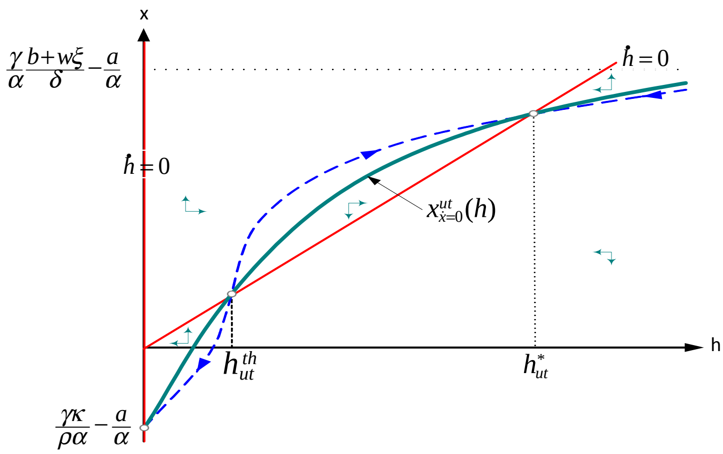

Figure 2: Case 2

Asymptotically the initially unskilled (and poor) completely lose their skill and income because both their economic and non-economic motive is small (small κ and small h,respectively). The initially sufficiently skilled converge to a high level h^* of human capital and income wh^* because their non-economic motive is strong (large b and h).

Proof. Corollary of Lemma 1. If $x_{\dot{x}=0}(0) < x_{\dot{h}=0}(0)$ and the $\dot{x} = 0$-isocline rises sufficiently fast compared with the $\dot{h} = 0$-isocline as in Figure 2, then the two functions of (5) have two strictly positive intersections, $h^{\text{th}} > 0$ and $h^* > h^{\text{th}}$. $x_{\dot{x}=0}(0) < x_{\dot{h}=0}(0)$ is satisfied if the economic motive is not strong ($\kappa < a\frac{\rho}{\gamma}$). The $\dot{x} = 0$-isocline rises sufficiently fast if the non-economic motive b is sufficiently strong $b > \left(\frac{\alpha}{\gamma}\rho + a\right)\frac{\delta}{\gamma}$. The arrows in Figure 2 indicate the direction of movement imposed by the FOC (4). The stationary solution at $h^{\text{th}} = 0$ is an unstable node and the two other stationary solutions at $h = 0$ and at $h^*$are saddle point stable. Among all paths satisfying (4) only those indicated by the dotted saddle path in Figure 3 satisfy the transversality condition. ■

Assuming a non-dominating economic motive ($\kappa < \min\{a\rho/\gamma, b\rho/\delta\}$), the threshold case occurs above a critical strength of the non-economic motive ($b = \left(2\sqrt{\alpha\left(a\frac{\rho}{\gamma} - \kappa\right)} + \alpha\frac{\rho}{\gamma} + a\right)\delta$), that increases with the agents's impatience ρ, the rate of depreciation δ and the cost of effort and decreases with rising effectiveness of effort.

Case 3, Weak economic and non-economic motives: Deterioration to passivity

Proposition 4 *If both the economic and the non-economic motives are weak, then no household exerts sufficient effort to raise its human capital. Formally: If $b\rho/\delta < \kappa < a\rho/\gamma$ or [$\kappa < \min\{a\rho/\gamma, b\rho/\delta\}$ and $b < \frac{2\delta\sqrt{\alpha\gamma(a\rho-\gamma\kappa)}+\alpha\rho\delta+a\delta\gamma}{\gamma^2}$], then $h_* = 0$ is the only non-negative real stationary solution and $\lim_{t\to\infty} h_t = h_* = 0$ for all h_0. See Figure 3.*

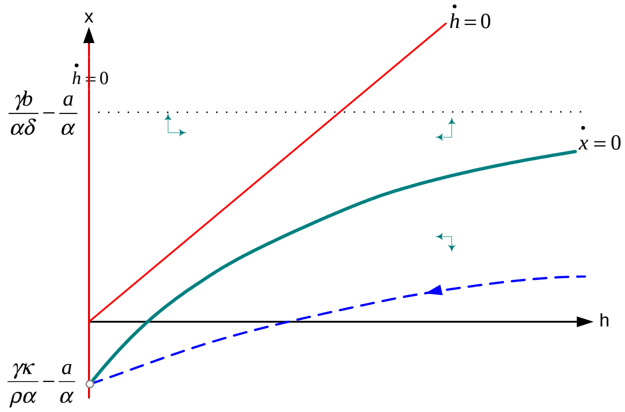

Figure 3: Case 3

The economic and the non-economic motives together are dominated by the cost of effort everywhere (even if h is large).

Proof. Corollary of Lemma 1. If $x_{\dot{x}=0}(0) < x_{\dot{h}=0}(0)$ and the $\dot{x} = 0$-isocline does not rise fast with h as in Figure 3 the two isoclines have no positive intersection. The trivial stationary solution at $h = 0$ is saddle point stable and among all paths satisfying (4) only those indicated by the dotted saddle path in Figure 3 satisfy the transversality condition. ■

Proposition 9 in Appendix 8.1 shows that if the non-economic motive is sufficiently strong, the three cases shown in 1 to 3 can be generated by a shift of the cost of effort a. Essentially it is shown that for sufficiently large b, if a is increased from a sufficiently low to a sufficiently high level, then the $\dot{x} = 0$-isocline is shifted down to generate the three figures 1, 2 and 3).

4 Polarization and simple transfers

Consider an economy with mass 1 of identical households (up to initial skill) that satisfy the conditions of Case 2 (Proposition 3). n^{u} agents start with human capital below the individual threshold ($h_0^i < h^{\mathrm{th}}$ for $i \in [0, n^{\mathrm{u}}]$) and $n^{\mathrm{s}} = 1 - n^{\mathrm{u}}$ households start above the threshold ($h_0^i > h^{\mathrm{th}}$ for $i \in [n^{\mathrm{u}}, 1]$). After a while the n^{s} households initially above the threshold will cluster in the neighborhood of the stable attracting steady state h^* while the others will have lost much of their initial human capital.

The inequality that occurs with such polarization would call for redistribution from rich to poor in most industrial countries. The present section studies the effect of a simple unconditional transfer on the behavior of the beneficiary. The conclusions of the present section do not depend on the motives of the funding party.

What happens if a household with low human capital and correspondingly low income is paid a transfer M ensuring a minimal standard of living?

Proposition 5 *If in the absence of transfers, an agent is in Case 1, then any transfer introduces a threshold level $h^{th}(M) > 0$, such that if initial skill $h_0 < h^{th}(M)$ the agent reduces his effort and will asymptotically lose all his skill. The agent will have enough to eat (his life-time utility at $t = 0$ is raised) but remains uneducated. If without transfer, the agent was already in Case 2, then the transfer raises the threshold ability $h^{th}(M)$ and raises it the more, the larger the transfer ($h^{th}(M)$ increasing).*

The transfer can therefore prevent economic *and* personal emancipation towards high skill and income. Unskilled but patient and motivated households, initially in Case 1 (Figure 4), which would have liberated themselves from poverty and low skill, are enticed to passivity and will remain unskilled in the presence of transfers.

Proof sketch. For simplicity assume that the group eligible for social transfer payments are exempted from any (further) tax.[5] The individual household problem (1) is therefore unchanged

[5] The model can for instance be closed by introducing a constant marginal labor income tax to balance government budget. The individual budget constraint becomes $c_t^i = T_t + wh_t^i = T_t + (1-\tau)\widetilde{w}h_t^i$, where $\widetilde{w}$ is the gross wage per labor efficiency unit, τ is the tax rate on labor income, $w = (1-\tau)\widetilde{w}$ is the after tax wage. Government budget is balanced

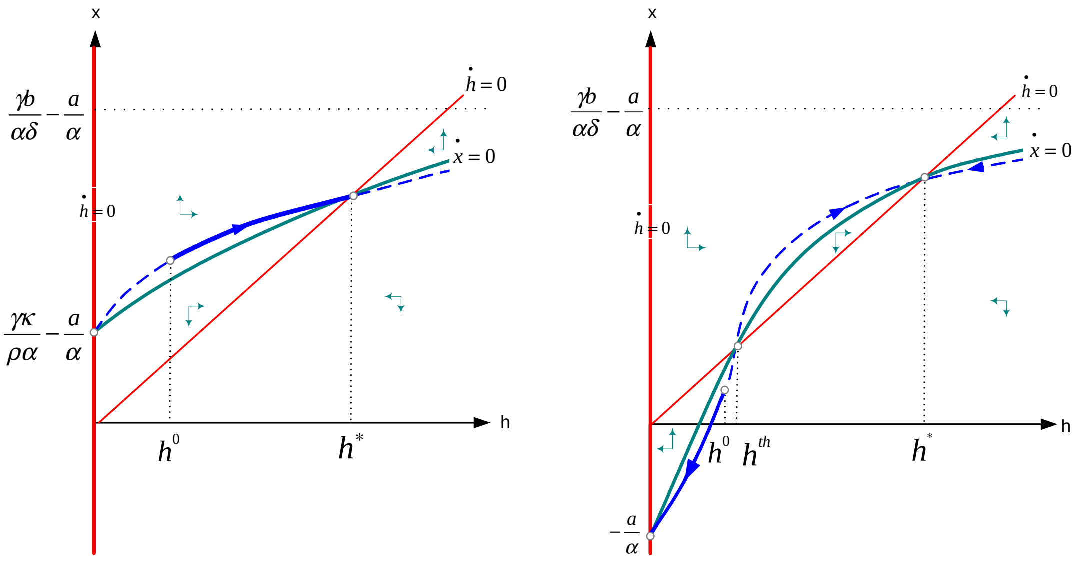

Figure 4: Left: Initially poor and motivated household Right: Same household receiving transfer.

except that $m(M + wh)$ replaces $m(wh)$. The dynamic system defined by the FOC and the rule of skill accumulation becomes

$$\begin{cases} \dot{x} = \frac{v_x(\rho+\delta h)-(m_h+b)\gamma h}{v_{xx}} = \frac{(a+\alpha x)(\rho+\delta h)-(\kappa \frac{w}{M+wh}+b)\gamma h}{\alpha} \\ \dot{h} = (\gamma x - \delta h)h \end{cases} \tag{9}$$

with isoclines

$$\begin{cases} \dot{x} = 0 \text{ if } x = x_{\dot{x}=0}(h, M) := \frac{\gamma}{\alpha} \frac{\kappa \overbrace{\frac{wh}{wh+M}}^{1 \text{ for } M=0} + bh}{\rho+\delta h} - \frac{a}{\alpha} \\ \dot{h} = 0 \text{ if } h = 0 \text{ or } x = \frac{\delta}{\gamma} h \end{cases}$$

The essential difference to the system without the transfer is that $x_{\dot{x}=0}(0, M) = -\frac{a}{\alpha} < 0 = x_{\dot{h}=0}(0)$ for any $M > 0$ independent of the strength of the economic and non-economic motives of effort. As a consequence Case 1 is not possible if $M > 0$. Recall that already in the baseline model (with $M = 0$) the non-economic motive was too weak to justify the effort necessary for activation at low h. Given the low absolute effectiveness of effort at small h ($d\dot{h}/dx = \gamma h \underset{h \to 0}{\longrightarrow} 0$), only a sufficiently strong economic motive could provide the incentive to raise h. Remember that a large m_h is needed to guarantee high effort (large $\dot{x} = 0$) if h is low. Without the Inada-condition guaranteeing $m_h = \frac{dm(wh)}{dc} \frac{dwh}{h} = \frac{\kappa}{h} \to \infty$ for $h \to 0$, even a large κ would not activate an unskilled household. With a transfer $M > 0$ the Inada condition looses its bite, since the marginal felicity of consumption $m_h = \frac{dm(wh+M)}{dc} \frac{d(wh+M)}{dh} = \kappa \frac{w}{wh+M} \to \kappa \frac{w}{M} > 0$ for $h \to 0$. The proposition is proven in more detail in Appendix 8.2. ■

if $\int_0^1 T_t^i di = \tau \widetilde{w} \int_0^1 h_t^i di$, so that if all other households are at steady state, then $T_t = \tau \widetilde{w}_t \int_0^1 h_t^i di$ is constant and the individual household problem is the same as before, where now $w = (1 - \tau)\widetilde{w}$.

A Samaritan's Dilemma Proposition 5 does not depend on the source of the transfers or on the motivation of the contributors. Suppose that, starting at a polarized equilibrium, a hypothetical social planner or a real government wants to raise both the disposable income and the skills of the poor. If simple transfers were the only available policy, this double aim of feeding and activating would establish an unsolvable dilemma for the government, a version of the "Samaritan's Dilemma": An unconditional transfer alleviates the material misery of the poor and unskilled but at the same time reduces their material motivation to exert effort and thus tends to perpetuate low skills and low earnings. Even worse, the low skill households that would actively raised themselves from poverty in the absence of the transfer will now be passive. The transfer can only mitigate the current misery of the poor by at the same time reducing the likelihood that they will become skilled and self-reliant.

5 Activating welfare

5.1 Pareto-improving activation

So far the amount of the (non-rival) public good (G_t) was exogenously given for individual households. Because G_t affects felicity in an additive separable way, the solution of the individual household problem (1) did not depend on the path of the public good ($\{G_t\}_{t\geq 0}$). Without a fiscal system organizing the funding of the public good, G_t would be zero for all $t \geq 0$. The market equilibrium then simply consists of the collection of solutions to the individual household problems of the previous section.

Assume that the economy has established a system to finance the provision of the public good, where individual contributions depend on personal income. In such an economy rich households gain from the education of the poor and may have an incentive to finance a transfer to the poor provided this raises their long-term income. As unconditioned transfers increase the passivity of the poor, they do not provide such an incentive. Starting from a polarized equilibrium, a welfare system able to bring about a Pareto-improvement has to guarantee

(a) **Activation**: The welfare system has to provide an incentive for the unskilled to exert enough effort to raise their human capital above the threshold.

(b) **Approval by the unskilled**: Activation could be forced by punishing for a *lack* of effort, but this would reduce the utility of the activated household. Thus, activation has to be sweetened by a transfer, which should be high enough to compensate the beneficiaries for their additional effort. This necessitates a transfer financed by the rest of the society.

(c) **Approval by the contributor**: While the transfer should be high enough to activate the unskilled, it should be low enough to be worth financing by the contributors. They have to be compensated by a sufficient increase of the public good (or reduction of their own tax bill) in the sufficiently near future.

As in Section 4, consider an economy with mass 1 of identical households (up to initial skill). A measure of n^{s} agents start with high initial skill $h_0^i = h_0^{\mathrm{s}} > 0$ for $i \in [n^{\mathrm{u}}, 1]$ and a measure of n^{u} agents start with low initial skill $h_0^i = h_0^{\mathrm{u}} < h_0^{\mathrm{s}}$ for $i \in [0, n^{\mathrm{u}}]$. I call such an allocation **persistently polarized** if $0 = \lim_{t\to\infty} h_t^{\mathrm{u}} < \lim_{t\to\infty} h_t^{\mathrm{s}}$ and $c_t^{\mathrm{u}} \leq w h_0^{\mathrm{u}}$ for all $t \geq 0$. Note that this definition does not depend on the economic environment under which the allocation occurs, e.g. on the the tax-scheme or public-good funding system that may be in effect.

The per capita cost of providing one unit of the public good is one unit of the consumption good. Furthermore, to focus attention on an environment in which the already activated skilled are willing to pay transfers for a while if this leads to the activation of the initially unskilled, I assume throughout this section, that the public good is sufficiently attractive (ξ sufficiently large) and that the initially skilled are sufficiently skilled (h_0^s sufficiently large). These two assumptions are quantified in Appendix 8.3.2 in terms of exogenous parameters (see (28) and (29)). The conditions are less restrictive than the corresponding conditions generating persistent polarization in the basic model (conditions of Proposition 3 guaranteeing existence of a threshold h^{th} and $h_0^{\mathrm{u}} < h^{th} < h_0^{\mathrm{s}}$).

Theorem 6 *Every feasible persistently polarized allocation* $z = \{c_t^i, x_t^i, h_t^i, G_t\}_{i\in\{u,s\},t\geq 0}$ *is Pareto-inefficient.*

Proof: Corollary of Theorem 7 below. ■

5.2 Implementation

5.2.1 Preview

Theorem 7 (Section 5.2.3) will show that there exists a tax-transfer function implementing a Pareto-improving allocation $\widetilde{z}$ upon any persistently polarized initial allocation. Before turning to a more formal statement and a complete proof of the theorem, I sketch how the proposed tax-transfer scheme satisfies the three above requirements (a)-(c):

(a) Activation. The tax-transfer scheme has to provide the incentive for the unskilled to exert enough effort to raise their human capital. The scheme proposed in Section 5.2.3 does so by raising the elasticity $\eta := \frac{dy_{\mathrm{dis}}^u(y)}{dy}\frac{y}{y_{\mathrm{dis}}^u(y)}$ of disposable income $y_{\mathrm{dis}}^u(y)$ with respect to pre-tax income $y = wh$ above 1. To see how this activates, remember that a rise in the marginal utility m_h, generated for instance by a rise in κ (the parameter measuring the economic motive), shifts up the $\dot{x} = 0$-isocline.

Suppose that κ is sufficiently small (and b sufficiently large) to generate the threshold case depicted in Figure 2. Raising κ (given all other parameters) would shift up the $\dot{x} = 0$-isocline, generating Figure 1a or, further raising κ, Figure 1b. A tax-transfer scheme can of course not manipulate the preference parameter κ. However, by raising the elasticity η of disposable income y^u_{dis} it generates exactly the same effect on m_h and hence on the FOC of intertemporal optimization: With $m(wh^u) = \kappa \log y^u_{\text{dis}}(wh^u)$, the marginal felicity of human capital becomes $m_h(wh^u) = \kappa\eta\frac{1}{h}$ instead of $m_h(wh^u) = \kappa\frac{1}{h^u}$ in the basic model: The curve $x^{\eta}_{\dot{x}=0}(h)$ in Figure 6 corresponding to a high elasticity η is identical to the $x_{\dot{x}=0}(h)$ curve in Figure 1b corresponding to $\eta = 1$ but a higher level of κ. As indicated in Figure 6 this activating scheme is in effect only for h^u that are smaller than some predetermined level $\widetilde{h}$.

(b) Approval by the (initially) unskilled. To warrant the approval of the initially unskilled, the proposed tax-transfer scheme has to provide them with a transfer as long as they are still very poor. Thus $y^u_{\text{dis}} > y^u$ for low h. Together with $\eta > 1$ (large m_h) this leads to a negative marginal tax: $1 - \frac{dy^u_{\text{dis}}(y^u)}{dy} = 1 - \eta\frac{y^u_{\text{dis}}(y^u)}{y^u} < 0$ if $\eta > 1$ and $y^u_{\text{dis}} > y^u$. Activation plus transfer thus requires a negative marginal income tax.

Note that the initial transfer to the unskilled need not be very large to grant their approval to an activating η if their initial h^u_0 is small (If not, wait with the beginning of the welfare program unit h^u_t is sufficiently small, which must eventually occur since at any polarized allocation $h^u_t \to 0$).

(c) Approval by the skilled. Together with the fact that the disposable income y^u_{dis} guaranteed in (b) can be relatively small, (a) creates a favorable environment also to satisfy (c): A large elasticity η of disposable income $y^u_{\text{dis}}(y)$ will make sure that the skill and pre-tax income y^u of the u is larger than a small guaranteed y^u_{dis} sufficiently soon. Once this is the case, the initial investment of the contributors can in principle start to pay a return. The proposed tax-transfer scheme will pay this return in terms of public good contributions, a currency which in the present setting is particularly valuable to rich contributors (large h^s) with a high valuation of the public good (large ξ) (since the marginal felicity of private consumption is declining, while that of a coordinated increase of the public good is not). The particular scheme proposed in Theorem 7 ceases to artificially force the initially unskilled agents into high effort, when this is no longer required. After a transitory period of transfers and activation, say when h^u has reached the predetermined level $\widetilde{h}$), the scheme switches to the (unique) equal-weight utilitarian policy. Section 5.2.2 formally defines equal-weight utilitarian welfare and shows that the utilitarian policy function $x^{\text{ut}}(h^i)$ always exhibits threshold dynamics. The auxiliary purely utilitarian $\dot{x} = 0$-isocline $x^{ut}_{\dot{x}=0}(h)$ and policy function $x^{\text{ut}}(h)$ of Figure 5 reappear in Figure 6, where they are relevant only for $h^u > \widetilde{h}$. Not to endanger activation, the Pareto-improving tax-transfer scheme of Theorem 7 only switches to implementing the utilitarian policy after the initially unskilled have overpassed the utilitarian threshold, i.e. we must have $\widetilde{h} > h^{\text{th}}_{\text{ut}}$. The skill of all agents converges to the same upper stationary state of the utilitarian policy (see Figures 5 and 6). Thus, in the long-run the tax-transfer scheme implements the unique symmetric first best allocation of an economy with initially identical agents.

5.2.2 Definition and auxiliary welfare concepts

This section formally defines the notion of implementation and solves the utilitarian welfare problem. To further prepare the complete proof of Theorem 7 it introduces a second auxiliary concept, intra-group symmetric optimum, which will serve as a benchmark for each group $k \in \{u, s\}$ to be beaten by the Pareto-improving allocation $\widetilde{z}$ and also describes the behavior of the s during the transitory period of activation.

Definition Consider an income tax function $M(h_t^i, h_0^i, \overline{h}_t^u, \overline{h}_t^s)$ which determines the tax (or transfer if $M < 0$) of for agent i at t as a function on i's individual income wh_t^i, of i's group $k \in \{s, u\}$ (which is determined by initial income wh_0^i), as well as on the average incomes $(w\overline{h}_t^u, w\overline{h}_t^s)$ of the two groups, which are exogenous to i. The tax does not directly depend on i's present or past effort.

The **individual optimization problem given a tax-function** $M(h_t, h_0, \overline{h}^u, \overline{h}^s)$ *and given i's expectation* $\{\overline{h}_t^u, \overline{h}_t^s, G_t\}_{t\geq 0}$ *is*

$$\begin{aligned} & \max_{\{x_t^i\}_{t=0}} \int_0^\infty e^{-\rho t} u(c_t^i, x_t^i, h_t^i, G_t) dt \\ \text{subject to } \dot{h}_t^i &= (\gamma x_t^i - \delta h_t^i) h_t^i \text{ for all } t \geq 0, \\ c_t^i &= wh_t^i - M(h_t^i, h_0^i, \overline{h}_t^u, \overline{h}_t^s) \text{ for all } t \geq 0, \\ & \text{given } h_0^i,\ \{\overline{h}_t^u, \overline{h}_t^s, G_t\}_{t\geq 0} \end{aligned} \tag{10}$$

A (rational expectation) **equilibrium** z^* **given a tax-function** $M(h, h_0, \overline{h}_t^u, \overline{h}_t^s)$ *solves (10) given correct expectations* $\{\overline{h}_t^u, \overline{h}_t^s, G_t\}_{t\geq 0}$ *for all i (i.e.* $h_t^{i*} = \overline{h}_t^k = \int_{j\in I_k} h_t^{j*} dj$ *for all* $i \in I_k$*,where* $k \in \{s, u\}$ *denotes i's peer group) and satisfies the government budget constraint* $\int_I M(h_t^i, h_0^i, \overline{h}_t^u, \overline{h}_t^s) di = G_t$ *for all* $t \geq 0$*.* **A tax function implements an allocation** $\widetilde{z}$ *if* $\widetilde{z}$ *is an equilibrium given this tax function.*

Note that $M(h, h_0, \overline{h}_t^u, \overline{h}_t^s)$ depends on h_0 to distinguish between initially unskilled and skilled agents. Alternatively, we can write $M^{k_i}(h, \overline{h}_t^u, \overline{h}_t^s) := M(h, h_0^i, \overline{h}_t^u, \overline{h}_t^s)$ where $k_i \in \{s, u\}$ is i's group defined by initial skill, or even shorter for given $\{\overline{h}_t^u, \overline{h}_t^s\}_{t\geq 0}$, $M_t^{k_i}(h) := M(h, h_0^i, \overline{h}_t^u, \overline{h}_t^s)$.

The (equal-weight) utilitarian planner problem The (equal-weight) **utilitarian planner problem** given $(h_0^i)_{i\in I}$ solves

$$\max_{\{(c_t^i,x_t^i)_{i\in I},G_t\}_{t\geq 0}} \int_{i\in I}\int_{t=T}^{\infty} e^{-\rho t}\left[\left(m(c_t^i)+\xi G_t\right)+\left(bh_t^i - v(x_t^i)\right)\right] dtdi$$

$$\text{subject to } \int_{i\in I} c_t^i di + G_t = w\int_{i\in I} h_t^i di \text{ for all } t\geq 0 \quad (11)$$

$$\dot{h}_t^i = (\gamma x_t^i - \delta h_t^i)h_t^i \text{ for all } t\geq 0,\ i\in I.$$

Appendix 8.3.1 shows that under the assumptions of sufficiently large h_0^s and ξ, the solution to this problem fixes consumption at the same constant value $c_t^i = \frac{\kappa}{\xi}$ for all agents. Correspondingly public good provision is $G_t = w\overline{h}_t - \frac{\kappa}{\xi}$. Furthermore (for sufficiently large h_s^0 and ξ), the 'dynamic problem' of choosing $\{x_t^i, h_t^i\}_{i,t}$ can be decomposed into independent optimization problems for each i (Case (ii) in Appendix 8.3.1). For each of these problems (e.g. for each i) there exists a threshold value $h_{\text{ut}}^{\text{th}} > 0$ and a $h_{\text{ut}}^* > h_{\text{ut}}^{\text{th}}$ such that $\lim_{t\to\infty} h_t = 0$ for all $h_0 < h_{\text{ut}}^{\text{th}}$ and $\lim_{t\to\infty} h_t = h_{\text{ut}}^*$ for all $h_0 > h_{\text{ut}}^{\text{th}}$ (see Figure 5). In particular, the two isoclines generated by the FOCs of (11) are

$$\begin{cases} \dot{x} = 0 \text{ if } x_{\dot{x}=0}^i(h^i) = \frac{\gamma}{\alpha}\frac{(b+w\xi)h^i}{\rho+\delta h^i} - \frac{a}{\alpha} \\ \dot{h} = 0 \text{ if } h=0 \text{ or } x_{\dot{h}=0}(h) := \frac{\delta}{\gamma}h \end{cases} \quad (12)$$

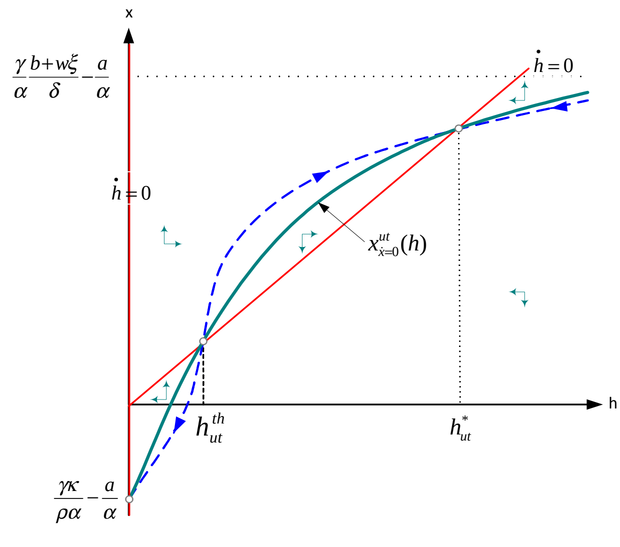

Figure 5: Utilitarian solution

When raising b or ξ, the $\dot{x}=0$-isocline (12) rotates upward (see Figure 5) , such that the individual threshold $h_{\text{ut}}^{\text{th}}$ (the inner stationary optimum h_{ut}^*) of the utilitarian problem is decreasing (increasing) not only with respect to the strength of the non-economic motive but also with respect to the preference for the public good.[6]

As has been noted, the allocation constructed in the proof of Theorem 7 will implement the equal-weight utilitarian solution after a transitory period of activation (i.e. at $t \geq T$). Applying the equal-weight utilitarian solution immediately (i.e. at $t \geq 0$ rather than at $t \geq T$) does not activate the unskilled (for small h_0^u) and can therefore not Pareto-improve over a persistently polarized initial allocation. See Section 6 for a further discussion of utilitarian welfare.

Implementing the (equal-weight) utilitarian solution Appendix 8.3.1 shows (among others) that, given $h_0^i = \overline{h}_0^k$ for all $i \in I_k$ and $n^s \overline{h}_0^s > \frac{\kappa}{w\xi}$, the utilitarian solution is implemented with the tax-transfer function

$$M^{k_i}(h_t^i, \overline{h}_t^u, \overline{h}_t^s) = wh_t^i - \frac{\kappa}{\xi} \cdot e^{\frac{\xi w}{\kappa}(h_t^i - \overline{h}_t^{k_i})} \tag{13}$$

At equilibrium with $h_t^i = \overline{h}_t^{k_i}$ this reduces to $M^{k_i}(h_t^i, \overline{h}_t^u, \overline{h}_t^s) = wh_t^i - \frac{\kappa}{\xi}$.

The intra-group symmetric solution A second auxiliary solution concept that will be used in the proof of Theorem 7 defines a benchmark z'^k for each group $k \in \{u, s\}$ to be beaten by the Pareto-improving allocation $\widetilde{z}$. From the perspective of the initially skilled (the s), the **best possible polarized allocation** $z^{s\prime}$ to be beaten by $\widetilde{z}$ is their preferred intra-group symmetric allocation given the maximal public good contribution $G_t^u = n^u w h_t^u$ by the initially unskilled. This allocation also describes the behavior of the s at $\widetilde{z}$ during the initial period of activation.

Intra-group symmetry requires $(c_t^i, x_t^i, h_t^i)_{t\geq 0} = (c_t^k, x_t^k, h_t^k)_{t\geq 0}$ for all $i \in I_k$, and $\frac{G_t^k}{n^k} = (wh_t^k - c_t^k)$.[7] The **intra-group symmetric solution** for group $k \in \{u, s\}$ given h_0^k and $\{G_t^l\}_{t\geq 0}$ for $k \neq l \in \{u, s\}$ solves

$$\begin{aligned} &\max_{\{c_t^k, x_t^k, G_t^k\}_{t\geq 0}} \int_{t=0}^{\infty} e^{-\rho t} \left[\left(m(c_t^k) + \xi(G_t^k + G_t^l \right) + (bh_t^k - v(x_t^k)) \right] dt di \\ &\text{subject to } c_t^k + G_t^k / n^k = wh_t^k,\ G_t^k + G_t^l \geq 0,\ G_t^k \geq 0 \text{ for all } t \geq 0 \\ &\dot{h}_t^k = (\gamma x_t^k - \delta h_t^k) h_t^k \text{ for all } t \geq 0. \end{aligned} \tag{14}$$

Note that this definition does not exclude the possibility that $G_t^l < 0$ for some t, in which case the constraint $G_t^k + G_t^l \geq 0$ implies, that the group k pays a transfer $-G_t^l$ to group l (each member of group k pays $-G_t^l/n^k$). Equivalently, if $G_t^l < 0$, then group k finance a total government budget of

[6] In the world without public good ($\xi = 0$ or small) the (utilitarian) threshold for group u is larger than the threshold in the individual problem of the baseline model if the skill of the other group h_0^s is sufficiently large (Case (i) in Appendix 8.3.1). However, even in the case that the planner decides to provide the public good (Case (ii) in Appendix 8.3.1), the threshold $h_{\text{ut}}^{\text{th}}$ of the (equal-weight) utilitarian solution can be larger than the threshold of the basic model.

[7] If each $i \in I$ gives up one unit of private consumption to finance the public good, then G grows by 1 unit. Thus, if each $i \in I_k$ contributes one unit of the private good G^k grows by only $n^k < 1$ units.

G_t^k, of which $-G_t^l$ is spent as transfers to the l and the remaining $G_t^k + G_t^l < G_t^k$ is spent on public goods.

The behavior of the agents of group k at their intra-group symmetric optimum is identical to the behavior at the utilitarian solution of the economy which replaces ξ by $n^s\xi$ and in which all agents start with the same initial skill h_0^k (see Lemma 14).

Implementing the intra-group symmetric solution Appendix 8.3.2 shows that given $h_0^i = \overline{h}_t^s$ for all $i \in I_s$ and $n^s\overline{h}_0^s > \frac{\kappa}{w\xi}$ the intra-group first best for the s is implemented by $M_{\text{sym}}^s(h_t^i, \overline{h}_t^u, \overline{h}_t^s) = wh_t^i - \frac{\kappa}{n^k\xi} \cdot e^{\frac{\xi w}{n^k\kappa}(h_t^i - \overline{h}_t^s)}$.

5.2.3 A Pareto-improving tax-transfer scheme

Theorem 7 *Consider a persistently polarized feasible allocation $z = \{c_t^i, x_t^i, h_t^i, G_t\}_{i\in\{u,s\},t\geq 0}$. There exists a tax-transfer function implementing a Pareto-improving allocation $\widetilde{z} = \{\widetilde{c}_t^i, \widetilde{x}_t^i, \widetilde{h}_t^i, \widetilde{G}_t\}_{i\in\{u,s\},t\geq 0}$ involving an initial transfer to the initially unskilled ($\widetilde{c}_t^u > w\widetilde{h}_t^u$ for small $t \geq 0$) and activation in the sense that all households cross a threshold level of human capital in finite time and converge to a high stationary level.*
Activation is achieved by a negative marginal tax of the initially unskilled u.
After the transitory period of activation the tax implements the (equal-weight) utilitarian solution, such that in the long-run the equilibrium converges to the unique symmetric first best allocation.

Note that the Pareto-improving allocations implemented by the proposed tax-transfer scheme need not be Pareto-optimal, an issue which is addressed in Section 6.

Proof:

Step (1): The benchmark (z'^s, z'^u) to be beaten. The benchmark (z'^s, z'^u) is chosen such that the agents of each group $k \in \{u, s\}$ prefer z'^k to every feasible persistently polarized allocation. For the s, the benchmark z'^s is the best intra-group symmetric allocation given $(G_t^{u\prime})_{t\geq 0}$ defined in (14).[8] For the u, the benchmark z'^u grants them the same path of skill as under the given polarized allocation without any effort ($h_t'^u = h_0^u$) but maximal possible consumption ($c_t'^u = wh_0^u \geq wh_t^u$) and provides them with any bounded sequence of public good provision.[9] Note that while (z'^s, z'^u) is *not* a feasible

[8] More precisely, $z'^s = (c'^s, x'^s, h'^s, G_t^{u\prime} + G_t^{s\prime})_{t\geq 0}$ where $G_t^{u\prime} = wh_0^u \geq wh_t^u$ and where $(c'^s, x'^s, h'^s, G_t^{s\prime})_{t\geq 0}$ is the best intra-group symmetric allocation given $(G_t^{u\prime})_{t\geq 0}$.

[9] More precisely, $z'^u = (c'^u, x'^u, h'^u, \overline{G}_t')_{t\geq 0}$ where $c_t'^u = wh_0^u \geq wh_t^u$, $h_t'^u = h_0^u \geq h_t^u$, $x_t'^u = 0 < x_t^u$ and where $(\overline{G}_t')_{t\geq 0}$ is any bounded sequence of public good provision. Note that in principle there may be a polarized allocation where the u exploit the s in the sense, that the s supply an unbounded amount of the public good ($G_t^s \to_t \infty$). This requires $h_t^s \to_t \infty$ and thus, $\lim_t x_t^s \geq \frac{\delta}{\gamma} \lim_t h_t^s = \infty$ as well as $v_x(x_t^s) = a + \xi x_t \to_t \infty$. Since at the same time the marginal utility is arbitrarily small for small h_0^u ($\lim_{h_0^u \to 0} m_c(wh_0^u) = \infty$), it is particularly easy to Pareto-improve such an allocation by

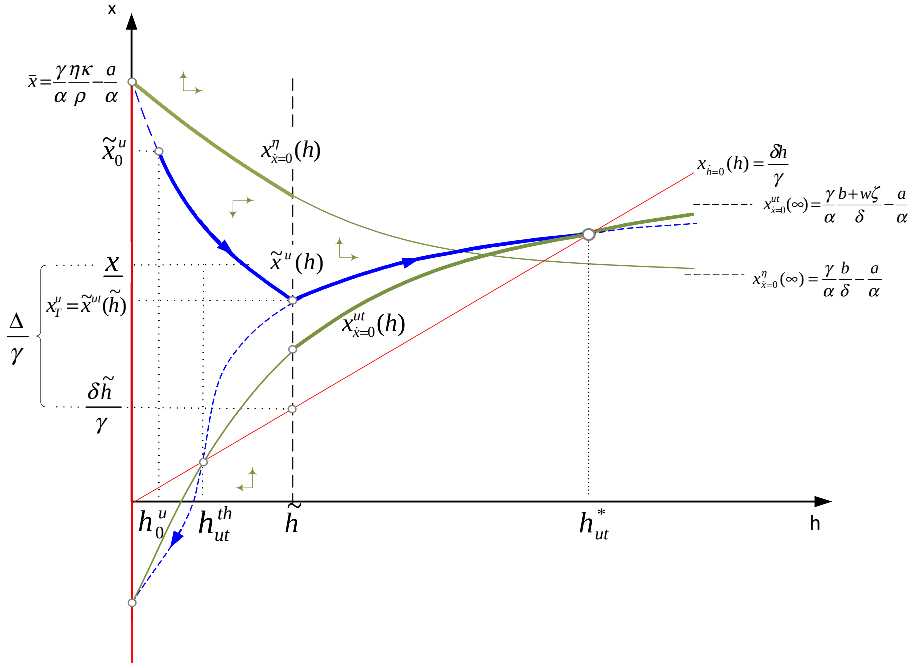

Figure 6: Implementation

allocation, a feasible allocation $\widetilde{z}$ Pareto-dominates any feasible polarized allocation if it dominates (z'^s, z'^u) in the sense that $\widetilde{z} \succ_s z'^s$ and $\widetilde{z} \succ_u z'^u$.

Step (2): Taxation and behavior

(2a) Taxation and behavior of the unskilled during the period of activation $t \in (0, T)$. Consider a low skilled agent in the threshold case of the economy without tax shown in Figure 2. The economic and the non-economic motives ($m(wh_t^u)$ and bh_t^u) together are not strong enough to activate an initially unskilled agent. To amplify the economic motive the tax-transfer scheme raises the marginal impact of skill on disposable income. More precisely, the proposed tax-transfer function

$$M_t^u(h_t^i) = wh_t^{\mathrm{u}} - y_t^{\mathrm{dis}}(h_t^{\mathrm{u}})$$

replaces gross income $y_t^{\mathrm{u}} = wh_t^{\mathrm{u}}$ by the new disposable income

$$y_t^{\mathrm{dis}}(h_t^{\mathrm{u}}) = w\phi q_t \cdot (h_t^{\mathrm{u}})^\eta \text{ for } t < T, \tag{15}$$

where ϕ is a constant and q_t is a time-autonomous term that does not depend on the agent's behavior and where the parameter $\eta > 1$ measures the elasticity of disposable income with respect to human capital. For $\eta > 1$, y^{dis} rises faster in h than does wh, which strengthens the economic motive for effort. The general form of the dynamic system summarizing the first order conditions of (10) is

reducing x_t^s and G_t^s and raising c_t^u.

identical to the dynamic system (4) of the basic optimization of Section 3. Replacing the untaxed income $y(h_t) = wh_t$ of the basic model by the disposable income (15), substitutes the marginal felicity of consumption $m_h = \frac{\kappa w\phi q_t \eta h_t^{\eta-1}}{w\phi q_t h_t^{\eta}} = \frac{\kappa\eta}{h_t}$ for $m_h = \frac{\kappa}{h_t}$ of the basic problem. Accordingly the fundamental equation of motion for x as well as the x-isocline $x^u_{\dot{x}=0}$ of the original household problem replaces κ by $\eta\kappa$:

$$x^{\eta}_{\dot{x}=0}(h_t) := \frac{\gamma}{\alpha}\frac{\eta\kappa + bh_t}{\rho + \delta h_t} - \frac{a}{\alpha} \text{ for } t < T. \tag{16}$$

For large η we always have $\frac{\gamma}{\alpha}\frac{\eta\kappa}{\rho} - \frac{a}{\alpha} > \frac{\gamma b}{\alpha\delta} - \frac{a}{\alpha}$ such that the phase diagram am (for $t < T$) takes the form of Figure 1b with a decreasing function $x^{\eta}_{\dot{x}=0}(h_t)$ (see Figure 6).

The activating tax-transfer scheme is relevant only for $t < T$. At $t = T$ the system switches to a tax-transfer scheme implementing the solution of the (equal-weight) utilitarian problem (see Step 2c). Thus for $t \geq T$ the agent follows the well defined saddle path determined by the utilitarian solution. Because instantaneous utility is concave in x the policy function is continuous at T: Given η, h^u_0, T, an initially unskilled agent chooses $\widetilde{x}^u_0$ such that the corresponding solution path of the transitory tax-transfer scheme reaches the saddle-path of the utilitarian regime exactly at T. Let $\widetilde{h} = \widetilde{h}^u_T$ be the corresponding human capital level. Raising η shifts up the $\dot{x} = 0$-isocline (see Figure 6) and the optimal effort $\widetilde{x}^u_t(h)$ for any given h. In particular, for η sufficiently large, $\widetilde{h} = \widetilde{h}^u_T > h^{\text{th}}_{\text{ut}}$ (details in Appendix 8.3.3).

Note that the marginal utility of consumption $m_h = \frac{\kappa\eta}{h_t}$ and therefore the dynamic system defined by individual optimization problem (10) do not depend on q_t. Hence, the time-autonomous term $\{q_t\}_{t\geq 0}$ does not affect the optimal effort and skill path such that we can determine q_t as a function of the optimal path

$$q_t = \frac{\overline{c}^u}{w\phi\left(\widetilde{h}^u_t\right)^{\eta}},$$

where $\overline{c}^u$ is a fixed level of consumption assigned to the u during the period of activation. Therefore, at individual optimum (with $h^{\text{u}}_t = \widetilde{h}^u_t$):

$$\widetilde{c}^u_t = w\phi q_t \cdot \left(\widetilde{h}^u_t\right)^{\eta} = \overline{c}^u \text{ for } t < T \tag{17}$$

and $M_t(h^i_t) = wh^{\text{u}}_t - w\phi q_t \cdot (h^{\text{u}}_t)^{\eta} = wh^{\text{u}}_t - \overline{c}^u \cdot \left(\frac{h^{\text{u}}_t}{\widetilde{h}^u_t}\right)^{\eta} = wh^{\text{u}}_t - \overline{c}^u$. The level of $\overline{c}^u$ is determined in Step 3 such as to make sure that both groups prefer $\widetilde{z}$ to z'.

Negative marginal tax for the initially unskilled during the period of transfers. To yield disposable income $y^{\text{dis}}(h^{\text{u}}_t) = w\phi q_t \cdot (h^{\text{u}}_t)^{\eta}$ the tax-transfer for the u must amount to

$$M^u_t(h^{\text{u}}_t) = wh^u_t - w\phi q_t \cdot (h^{\text{u}}_t)^{\eta} \text{ for } t < T$$

where $q_t = \frac{\bar{c}^u}{w\phi(\widetilde{h}_t^u)^\eta}$. The corresponding marginal tax is $\frac{dM_t(h_t^{\mathrm{u}})}{dh} = w\left(1 - \eta\phi q_t\,(h_t^u)^{\eta-1}\right)$. Since along the optimal path $h_t^u = \widetilde{h}_t^u$, the marginal tax along this path is

$$\frac{dM_t(h_t^{\mathrm{u}})}{dh} = w\left(1 - \eta\frac{\bar{c}^u}{wh_t^u}\right) < 0$$

as long as the u receive a transfer ($\bar{c}^u > wh_t^u$) and possibly longer since $\eta > 1$.

(2b) Taxation and behavior of the s during the transitory period. The tax for the s at $t \in (0, T)$ implements the solution of their intra-group symmetric problem (14) given $(\widetilde{G}_t^u = w\widetilde{h}_t^u - \bar{c}^u)_{t\geq 0}$ as described in Section 5.2.2 and taking into account that at $t \geq T$, the tax-system implements the (equal-weight) utilitarian solution.

Note that each agent in group s has to pay a transfer for small t ($\widetilde{G}_t^u = w\widetilde{h}_t^u - \bar{c}^u < 0$) and benefit from the fact that also the u contribute to the public good for larger t ($\widetilde{G}_t^u = w\widetilde{h}_t^u - \bar{c}^u > 0$ for t close to T). This has no effect on their intra-group optimal $\{x_t'^s, h_t'^s\}_t$ as long as $n^s\overline{h}_t^s > \frac{\kappa}{w\xi} + n^u(\bar{c}^u - wh_t^u)$. Since $\bar{c}^u$ is small for sufficiently small h_0^u this condition is satisfied if $n^s\overline{h}_t^s > \frac{\kappa}{w\xi}$ as in the case without transfer.

(2c) Taxation and behavior after the transitory period of activation. The tax for both groups at $t \geq T$ implements the utilitarian as described in Section 5.2.2. Since $\widetilde{h} > h_{\mathrm{ut}}^{\mathrm{th}}$, the u remain activated. Depending on the skills $(\widetilde{h}_T^s, \widetilde{h}_T^u)$ at T, the individual paths of effort at utilitarian optimum will in general differ for the two groups. In the long-run however, skill and effort converges to the unique symmetric first best allocation of an egalitarian society (remember that apart from initial skill, all agents share the same characteristica).

Step 3. Approval by the contributors

(3a) By construction, the proposed allocation $\widetilde{z}$ switches to the utilitarian solution as soon as h_t^u reaches a predetermined level $\widetilde{h} > h_{\mathrm{ut}}^{\mathrm{th}}$ at T. Compared to the benchmark allocation $(z_t'^s)_{t>T}$, the utilitarian solution $(\widetilde{z}_t)_{t>T}$ imposes additional effort on the s (as it internalizes the pubic good externality on the u) and provides them with more public goods (including public good provision by the u). Obviously, for $h_T^u = h_T^s$ the utilitarian solution (then unique first best symmetric allocation) is better for the s then the benchmark. By continuity it follows that this is also true if $h_T^u = \widetilde{h}$ is sufficiently close to h_T^s, which is satisfied for sufficiently large η. This guarantees that $\{\widetilde{z}_t^s\}_{t\geq T} \succ_s \{z_t'^s\}_{t\geq T}$.

(3b) For $\widetilde{z} \succ_s z'^s$ it remains to make sure that $(\widetilde{z}_t^s)_{t<T} \succ_s (z_t'^s)_{t<T}$. Since $\{\widetilde{c}_t^s, \widetilde{x}_t^s, \widetilde{h}_t^s\}_{t<T} = \{c_t'^s, x_t'^s, h_t'^s\}_{t<T}$ and felicity is linear with respect to $(G_t)_t$ this amounts to guarantee that the present value of public good provision till T is larger under $\widetilde{z}$ than under z' (i.e. $\int_0^T e^{-\rho t}\widetilde{G}_t dt > \int_0^T e^{-\rho t}G_t' dt$). Because $\{\widetilde{c}_t^s, \widetilde{h}_t^s\}_{t<T} = \{c_t'^s, h_t'^s\}_{t<T}$, the contribution of the s to the public budget (G_t^s plus transfers to the s) is also the same under the two regimes such that

$$\int_0^T e^{-\rho t}\widetilde{G}_t^u dt > \int_0^T e^{-\rho t}G_t'^u dt \tag{18}$$

or

$$\int_0^T e^{-\rho t}(w\widetilde{h}_t^u - \widetilde{c}^u)dt > \int_0^T e^{-\rho t}(wh_t^{\prime u} - c_t^{\prime u})dt$$

is sufficient for $\widetilde{z}^s \succ_s z'^s$. Note that the u will only be willing to activate if in exchange they receive a transfer for small t when their pre-transfer income still is very small. This means that their initial contribution to public good provision $\widetilde{G}_t^u = w\widetilde{h}_t^u - \widetilde{c}^u$ is negative. This initial reshuffling of public expenditures from public good provision toward social welfare is attractive for the s if $\widetilde{h}_t^u$ rises sufficiently fast to turn the initial transfer into a positive net contribution soon enough to satisfy Condition (18). Appendix 8.3.3(3b) shows that one can always choose η high enough to guarantee that the speed of skill accumulation $\widehat{\widetilde{h}}_t^u$ is larger than the discount rate ρ. A sufficient condition for (18) than is that the private consumption $\widetilde{c}^u = \overline{c}^u$ (specified in Equation (35)) assigned to the u during the transitory period of activation is not too generous.

Step 4. Approval by the beneficiary. If h_0^u is very small, consumption $c_t^u \leq wh_0^u$ at the polarized allocation z is very small as well. The marginal utility of consumption and therefore the benefit of even a small transfer is very large. More precisely, the appendix shows that $\lim_{h_0^u \to 0} \frac{\ln \overline{c}^u}{\ln c_t^u} = \infty$.

On the other hand, given η, effort is bounded by $\overline{x} := \frac{\gamma}{\alpha}\frac{\eta\kappa}{\rho} - \frac{a}{\alpha}$, such that the cost of effort is bounded by $v(\overline{x})$ for all $t \geq 0$. Furthermore $\overline{G}_t^u - \widetilde{G}_t^u$ is bounded for any t (because $\left(\overline{G}_t^u\right)_{t\geq 0}$ is bounded). Therefore $\widetilde{z}^u \succ_u z'^u$ for small h_0^u. If initially, h_0^u is not sufficiently small, the beginning of the transfer payment is delayed accordingly. ■

6 Discussion and Extensions

6.1 Discussion of the assumptions·

1. **Human capital**. While the variable "years of education" satisfies the basic requirements the present paper imposes on human capital (it generates income, is a direct source of utility, and is acquired with effort),[10] the term human capital is used here in a broader sense, inspired by psychology literature concerned with human needs as determinants of behavior and subjective well-being (SWB). Summarizing previous empirical work on Maslow's influential hierarchical

[10] Studies by the US Census Bureau and many other agencies have consistently shown that people with a higher level of education earn more money than those with less education and it has also been shown that this higher level of annual earnings translates into significant increases in overall lifetime earnings. Furthermore, Witter et al. [1984] "found that education is significantly positively related to positive subjective well-being (SWB). While the contribution of education to SWB is relatively small in Witter et al. [1984], Blanchflower and Oswald [2004] find that the non-economic variables in happiness equations, in particular education enter with large coefficients, relative to that of income. In particular, "education is playing a role independently on income."

theory of needs [Maslow 1943],[11] Mitchell and Moudgill [1974] propose a two-step hierarchy of human needs. Similarly, Wahba and Bridwell [1976] find "that a dual level hierarchy of needs may provide a viable alternative to Maslow's multilevel need hierarchy." They suggest that human needs can be categorized as either maintenance needs or growth needs (including mastery, respect, self-direction and autonomy). Diener and Tay [2011] conduct a comprehensive and globally representative empirical study on needs and SWB and find strong evidence that the fulfillment of a list of needs including the above-mentioned maintenance and growth needs derived from Maslow [1943] is closely and consistently associated with SWB. Using the terms of Wahba and Bridwell [1976], consumption of the present article satisfies maintenance needs while the state variable human capital satisfied growth needs.

2. **Specific class of felicity functions**

   (a) **Additive separability.** Diener and Tay [2011] find "substantial independence in the effects of needs on SWB" and conclude "that the individual-level needs were primarily additive in their association with SWB." This is reflected in the additive separability of u with respect to c and h. The additive separability with respect to effort is standard in models where effort, leisure or hours worked enter the felicity function. To otherwise keep the influence of x and G as simple as possible, they also enter instantaneous utility in an additively separable way.

   (b) **The hierarchy of needs**. A central aspect of Maslow's hierarchical theory of needs (Maslow [1943]) was the deprivation/domination proposition, which hypothesized that the deprivation of an important need will lead to the domination of this need. Wahba and Bridwell [1976] review evidence indicating that the deprivation/domination proposition is relevant when maintenance needs, but not when growth needs, are deprived. An elegant way to phrase these results in standard terms of economic modeling and consistent with (a) is to require that $m(c)$ satisfies the Inada-condition for small c ($\lim_{c \to 0} m'(c) = \infty$) with $m'(c) > 0$, $m''(c) < 0$, while $f'(h)$ is neither particularly large for small h nor decreasing with h. This is formulation is also compatible with Diener and Tay [2011], who find that although different needs affect SWB in an additive way, "people tend to achieve maintenance needs before other needs." The simplest functional form satisfying these requirements is (2) with $m(c) = \kappa \ln c$ (where κ measures the strength of the economic motive for effort) and $f(h) = b \cdot h$ (where b measures the strength of the non-economic motive of effort).

[11] In a survey of significant contributions to management literature, Matteson [1974] ranks Maslow's article second among 1,694 total article citations and Roberts [1972] already lists 140 authors referring to Maslow's theory of human needs.

3. **Income and expenditures.** The assumption $c = wh$ combines two strong simplifications:

   (a) **Perfect substitutes.** First, individual productivity and income wh are proportional to human capital at the exogenous rate w. As in the theories of Loury [1981], Becker and Tomes [1979], and much subsequent literature, human capital is reduced to a homogeneous variable h measured in terms of the same unit of efficiency for all agents. Different levels of human capital are perfect substitutes in the production of output. Apart from simplifying the model, this excludes one source of persistent inequality which arises when a strictly positive supply of *different* types of human capital (occupations) is essential for producing any output (see for instance Mookherjee and Ray [2003]).

   (b) **Absence of physical capital and credit markets.** Second, consumption c equals income. Financial markets are completely absent from our model, households can neither save nor dissave. This substantially simplifies studying the complete dynamics of the optimization problem. In view of the literature on credit constraints and inequality initiated by Galor and Zeira [1993], it is important to emphasize that self-financing constraints are *not* the reason for the multiplicity of stationary solutions in the present setting (see Section 6.2, Proposition 8).[12]

4. **Accumulation.** $\widehat{h} = \gamma x - \delta h$:

   (a) **The effectiveness of effort.** As in models including learning by doing, the current level of human capital in each period is determined by the sum of the past depreciated efforts of the agent. In other words, the growth rate of h depends positively on the current expended effort. An essential assumption for the existence of multiple stationary optima is that the effectiveness dg/dx of effort in raising human capital does not decrease with a rising h. The simplest functional form satisfying this requirement is (3), which assumes that the effectiveness γ of effort is independent from h.[13]

[12] For the distributional issues at the heart of much of this literature as well as of the present paper, it is essential that the multiplicity occurs at the level of individual optimization. In the present paper, the optimization problem of one person has multiple optimal steady states *given* the behavior of others. There is much literature where multiplicity of stationary solutions only occurs on an *aggregate* level (e.g., Skiba [1978], Davidson and Harris [1981], Dechert and Nishimura [1983], Ladron-de-Guevara et al.[1999]) or at the level of local interactions within neighborhoods or regions (e.g., Durlauf (1996), Benabou (1996), and Galor and Tsiddon (1997)).

[13] Note that this entails that the absolute effectiveness $d\dot{h}/dx = \gamma h$ of effort increases with the skill level. Formally equivalent one could assume that effort is the less unpleasant, the more one knows: $u(c, \frac{\text{effort}}{h}, h, G)$ and $\dot{h} = \gamma \cdot \text{effort}$ $-\delta h^2$. If we define $x := \frac{\text{effort}}{h}$ as relative effort, this leads to the maximization problem (1).

(b) **Depreciation.** The depreciation rate δh is an increasing (rather than constant) function of h merely to exclude the possibility of unbounded growth. The threshold dynamics also occur with the more standard accumulation rule. The difference to the present formulation would be that the strictly positive attractor of h would would be bounded in the case of a household that starts above the threshold, while it is bounded here.

6.2 Introducing a credit market

Consider the base line model of Section 3. A strong assumption of the basic maximization problem (1) is that, in every period, each agent consumes his entire labor income. In particular, an unskilled and poor agent is not able to borrow money today, even when this allows him to accumulate skill and raise his income. This assumption is not decisive for the possibility of threshold dynamics. Consider an economy with agents $i \in [0,1]$ and with financial wealth. The individual optimization problem of agent i adds a second state variable $k_t^i \gtreqless 0$ to (1) and replaces the constraint $c_t^i = wh_t^i$ with the constraint $\dot{k}_t^i = r_t k_t^i + wh_t^i - c_t^i$, where r_t is the interest rate. The initial state now is an $h_0^i \geq 0$ as before and in addition a $k_0^i \gtreqless 0$. To concentrate on the simplest case, assume that $\int_0^1 k_t^i di = 0$. Agents may differ in their parameters except for the discount rate.

Proposition 8 *A stationary equilibrium of the original economy without credit markets remains a stationary equilibrium in the economy with financial wealth and without borrowing constraints, where in addition $k_t^i = 0$ for all i and $r_t = \rho$.*

Proof. Consider a stationary equilibrium of the original economy with only one state variable (h). For each agent (index omitted) the constant values for x, h, c solve the three equations $0 = \dot{x} = \frac{\rho v_x - [(\rho+d)\phi + \gamma m_h + \gamma f_h]h}{v_{xx}}$, $0 = \dot{h} = (\gamma x - d)h$ (stationary solution to (4)) and $wh = c$. Second consider the optimization problem in the new economy with two state variables (h, k). The first order condition for consumption c together with the adjoint equation for the costate variable of k yield the usual Euler condition $\widehat{c}_t = r_t - \rho$. Thanks to the perfect credit market, the agent is able to perfectly smooth his consumption. Given $r_t = \rho$ it is optimal to do so, i.e. to choose a constant c. For any given c, the adjoint equation for the costate variable of h together with the accumulation rule for h exactly define the dynamic system (4) (taking into account that $m_h = \frac{dm(c(h))}{dc}\frac{dc}{dh} = wm_c$). Stationary k requires $0 = \dot{k}_t = r_t k_t + wh_t - c_t$, or, with constant $h_t = h, c_t = c$, $k_t = 0$, and $r_t = \rho$ that $wh = c$. Thus for initial financial wealth $k_0 = 0$, the stationary optima x, h, c to the problem with financial wealth have to solve the same three equations of the problem without financial wealth. ∎

Proposition 8 shows that the set of stationary optima is robust with respect to the introduction of financial wealth and the removal of borrowing constraints (given $k_0^i = 0$ for all i). The treatment of the complete dynamics with two state variables goes beyond the scope of the present paper. It would add an interesting second dimension: The interplay between financial wealth and human capital. The complete set of stationary optima in this economy (even given constant $r_t = \rho$) is of course much larger than the subset described in the proposition, since it is not restricted to zero initial financial wealth for the individual agent. Most importantly, one can show that the possession of initial wealth $k_0^i > 0$ — far from eliminating the path dependence of individual behavior — reinforces it. Higher initial wealth reduces the economic motive of an unskilled household to invest (non-monetary) effort to raise its human capital. Case 1 is then excluded. In fact, the effect of strictly positive initial financial wealth on the behavior of unskilled agents is very similar to the effect of a monetary transfer studied in Section 4.

6.3 Pareto-optimal Pareto-improvement, talent-slavery and income rank reversal

The tax-transfer system advocated in the proof of Theorem 7 *generates* long-run equality between agents that only differ in their initial skill endowment. It does so by raising both the present value utility U_0^i of every individual (it induces a Pareto-improvement upon the initial intertemporal allocation) and the long-run felicity u_t^i of every individual as well. However, the allocation implemented in the proof of Theorem 7 is in general not (ex ante) Pareto-optimal.[14] A natural question therefore arises: What would a **Pareto-optimal Pareto-improvement** over the polarized allocation look like (and which tax-function would implement such an allocation)?

Equal-weight utilitarian welfare freezes social class affiliation and involves talent slavery of the initially skilled. As has been noted, the (equal-weight) utilitarian allocation (applied at $t \geq 0$ rather than at $t \geq T$) is Pareto-optimal but does not Pareto-improve over a polarized allocation. For $h_0^u < h_{\text{ut}}^{\text{th}}$, the Samaritan's Dilemma is "solved" in favor of disactivating transfer. While the allocation constructed in the proof of Theorem 7 provides transitory transfers to the poor only at the cost of additional effort, the (equal-weight) utilitarian planner provides immediate and persistent transfers *and* comfort. Aiming at narrowing the distance between ex ante (dynastic) utilities (U_0^i), the equal-weight utilitarian solution freezes initial social class affiliation and raises the distance between classes in terms of skill, economic capacity and long-run felicity. For the initially skilled this means that instead of paying a transfer only for a transitory period to enjoy higher public good provision in the future, they have to pay perpetual transfers without ever getting anything in return. Worse still, they are persistently forced to exert higher effort and sustain higher skill than under their symmetric

[14] In particular, at $t = T$ the ratio c_t^u/c_t^s of consumptions of the two groups jumps up from $c_t^u/c_t^s = n^s\xi\bar{c}^u/\kappa$ to $c_t^u/c_t^s = 1$. In contrast Pareto-optimality requires constant c_t^u/c_t^s.

intra-group best solution. This is a form of what has been called talent slavery[15] in the literature and which has been rejected by political philosophers and legal scholars alike.[16]

General utilitarian welfare, if it activates the initially unskilled involves income rank reversal in the long-run. If an initially polarized society wants to overcome activation in a way approved by all involved groups it cannot resort to the equal-weight utilitarian solution. In the search for a Pareto-optimal Pareto-improvement over a polarized allocation it is natural then, to turn to the more general utilitarian problem of maximizing the weighted sum of individual utility for all vector of weights in the unit simplex (This problem is formally defined in Appendix 8.3.1 as it allows to subsume the equal-weight utilitarian problem and the intra-group symmetric problem under a same maximization problem). A companion paper (Funk [2015]) deals with the set of utilitarian welfare optima in more detail.

The smaller an agent's weight, say the u's weight p^u, the smaller her share in total consumption, the larger her p-optimal effort (given h_t^u) and her upper stationary $h^*(p^u)$ and the smaller the threshold $h_{\text{th}}(p^u)$ (with $\lim_{p^k \to 0} h_{\text{th}}(p^u) = 0$). In particular, the u can be activated for any $h_0^u > 0$ by choosing a sufficiently small p^u guaranteeing $h_{\text{th}}(p^u) < h_0^u$. Thus, since the solution to any p-planner-problem is Pareto-optimal, an activating Pareto-optimal allocation always exists. However, in contrast to the allocation advocated in Section 5.2.3, a utilitarian solution assigning a small weight p^u to the u, necessary to activate them when they are still unskilled, will continue to do so when h_t^u has grown beyond the threshold h^{th} of section 2, beyond the threshold of the equal-weight utilitarian solution $h_{\text{ut}}^{\text{th}}$ and will even continue to further activate the u when their skill h_t^u has overtaken the skill h_t^s of the initially high skilled s. In the long-run, the u are forced to exert higher effort and sustain larger skill than the s ($x^*(p^u) > x^*(p^s)$ for $p^u < p^s$). At the same time, a small p^u entails low consumption not only initially (when c^u is essentially financed by the s) but persistently. Thus after a transitory period of catching up, the u will for ever consume less than the s while they must persistently exert more effort and generate more income. In the long-run the corresponding utilitarian solution entails a persistent reversal of pre-tax and post-tax income ordering. Activating utilitarian redistribution persistently reverses the cross-sectional ranking of disposable income (the larger an agent's pre-tax income, the smaller her after tax income).

Furthermore, it is not guaranteed that a there exists a utilitarian optimum which Pareto-dominates

[15] In the present economy with two groups I say that agent suffers from talent slavery, if she is forced to persistently exert higher effort and sustain higher skill than under her symmetric intra-group best solution without benefitting from higher consumption (of private or public goods).

[16] See for instance Weinzierl (2014) and the references there. Most notably, already Rawls (1971) critizises utilirarism for its tendency to "force the more able into those occupations in which earnings were high enough for them to pay off the tax in the required period of time".

the Pareto-inefficient allocation of Section 5.2.3. In fact, Funk [2015] also shows, that the set of utilitarian welfare optima in general strictly smaller than the set of Pareto-Optima.[17] Funk [2015] therefore introduces a more general welfare criterion. Although the set of corresponding optima contains allocations neither contained in the set of utilitarian optima nor Pareto-dominated by any utilitarian optimum, the conclusions about talent slavery and income rank reversal remain the same: If the initial distribution of skill is sufficiently dispersed, than all corresponding optima are subject to talent slavery or income rank reversal.

7 Conclusion

As Gary Becker (Becker [1993]) already noted, ignoring human capital as a direct source of satisfaction or more generally, as a source of non-economic returns, quantitatively underestimates the total return of investment in human capital and therefore underestimates the motivation to invest. This paper has shown that it also alters the *qualitative* nature of the interaction of human capital, the investment in human capital and the *economic* return of human capital. Imposing the standard Inada conditions for consumption, but not for human capital – an assumption in line with psychology literature on subjective well-being – the intertemporal individual household problem has multiple solutions and threshold dynamics occur when the economic motive for effort is too weak to activate the agent for low levels of human capital and while the non-economic motive for effort remains strong for high levels of human capital. The critical strength of the non-economic motive above which the threshold case occurs increases with the agents's impatience, the rate of depreciation and the cost of effort and decreases with rising effectiveness of effort.

The fact that the multiplicity of solutions originates at the level of individual optimization rather than through the interaction between individuals is crucial for the distributional issues at the center of this paper. Its occurrence on the individual level makes it a source of inequality or polarization within a society. It is the fact that the individual faces a threshold given the behavior of others which makes loans or simple transfers ineffective and conditional transfers (or a negative marginal income taxes) effective in activating the individual.

In an economy without externality, any market equilibria – segregated or not – is Pareto-optimal. The unskilled only remain unskilled when this is optimal for them. And the rich have no incentive

[17] This applies in particular for or small h_0^u and small p^u (the relevant case for activating agents with low initial skill). For small p^u the sufficient condition guaranteeing a continuous policy function $x(h)$ in the present setting (see Appendix 8.1) is violated. The companion paper gives sufficient conditions under which the p-optimal policy function is not continuous (and not uniquely defined) at the threshold $h^{\text{th}}(p^u)$, which in turn leads to a non-convex set of feasible pairs (U_0^u, U_0^s) of present value utility given the initial h_0.

to pay a transfer (conditional or not). However even in the absence of externalities, redistribution – although not Pareto-improving – will evoke less resistance by the contributors if conditioned on activation, since this limits their transfer payments to a transitory period. A policy that both feeds *and* activates finds a compromise between beneficiary and contributors by reducing the future number of the needy.

The case for conditional redistribution is more compelling even when there is a sufficiently strong common interest in the education of the poor. As we have seen, an appropriate activating welfare scheme then Pareto-improves upon any persistently polarized allocation. In the present paper such a common interest arises due to the presence of public goods that are the easier to finance as the number of skilled and contributing households increase. The conclusions concerning Pareto-improving activating welfare remain valid if the material advantage the rich derive from the education of the poor is replaced (or reinforced) by more paternalistic motives (a dislike for extreme poverty, for low education, or for lost opportunities, or, more specifically, a willingness to support those that try to help themselves).

The concrete tax-transfer scheme proposed in the proof of Theorem 7 describes an activating Pareto-improving policy but does not lead to a Pareto-optimal allocation. In fact, we have seen that Pareto-optimal activation leads to a negative cross-sectional relation between earned income and after tax income. The concrete tax-transfer scheme, by generating long-run equality between agents that only differ in their initial skill endowment, solves the conflict between ex-ante Pareto-optimality and exclusion of persistent talent slavery in favor of the latter.

The paper has emphasized the possibility of persistent inequality among inherently equal households. The poor are poor due to unfavorable initial conditions. Granting the importance of inherent heterogeneity in reality does not much alter the conclusions: If *within* a group of otherwise identical households not all are initially positioned at the same side of the threshold, then this group will eventually become completely segregated into two separate subgroups, one relatively unskilled and poor and one relatively skilled and rich. It is true that in a world with (inherently) heterogeneous agents the argument for conditional transfers applies even when all individual optimization problems have unique stationary solutions: Strongly motivated and skilled may be willing to pay for the activation of less motivated types because this increases the future funding of public goods. However, without multiplicity of stationary optima, such transfers have to be paid permanently, since the unskilled are unskilled for immutable reasons when households mainly differ by fixed characteristics (their types).[18] In contrast, when human capital is a state variable that can be changed over time, a low-skill trap, even if individually optimal, can in principle be permanently surmounted. It is the possibility of rais-

[18] A similar remark applies to the static theory of optimal taxation.

ing an individual household's human capital above a threshold required to activate it *once and for all* that makes activating welfare particularly attractive.

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8 Appendix

8.1 Appendix: The Individual Optimization Problem, Proof of Lemma 1.

Stability of the interior stationary solutions to (4) The determinant of associated Jacobian of the system (6) at the non-zero stationary solutions is $\det J(h_{1,2}) = -\left[\pm\frac{\sqrt{(\gamma^2 b-\alpha\rho\delta-a\delta\gamma)^2-4\alpha\delta^2\gamma(a\rho-\gamma\kappa)}}{\alpha}\right]h$, where $\det J(h^{\text{th}}) > 0$ and $\det J(h^*) < 0$. Therefore h^{th} is unstable (whenever it exists, thus in Case 2) and h^* is saddle-point stable (whenever it exists, thus in Cases 1 and 2).

Stability of the trivial stationary solution to (4) The determinant of the associated Jacobian of the system (6) at the steady state with zero ability ($h_* = 0, x_*$) is $\det J(h_*) = \gamma\rho\left(\frac{\gamma}{\alpha}\frac{\kappa}{\rho} - \frac{a}{\alpha}\right)$. In Case 1 $\det J(h_*) > 0$ and the trivial steady state (h_*, x_*) is unstable and in Cases 2 and 3 $\det J(h_*) < 0$ such that the trivial steady state $h_* = 0$ is a saddle point.

The transversality condition The necessary transversality condition for (1) is $\lim_{t\to\infty} e^{-\rho t}\lambda_t h_t = 0$. Inserting the FOC $\lim_{t\to\infty} e^{-\rho t}\frac{v_x}{\gamma h}h_t = \lim_{t\to\infty} e^{-\rho t}\frac{v_x}{\gamma} = \lim_{t\to\infty} e^{-\rho t}\frac{a+\alpha x}{\gamma} = \lim_{t\to\infty} e^{-\rho t}\frac{\alpha x}{\gamma}$. Any path satisfying (4) and converging to one of the three possible stationary solutions (including these stationary solutions) satisfy the transversality condition $\lim_{t\to\infty} e^{-\rho t}\frac{\alpha x}{\gamma} = 0$ since in all these cases $\{|x_t|\}_t$ is bounded. In contrast, it can be shown that any path starting off the proposed policy function satisfying (4) violates the transversality condition.

Sufficient conditions Hartl et al. [2004] show that if the unstable stationary solution lies in the concave domain of the Hamiltonian, then this stationary solution is a node of the dynamic system and the policy function $x(h)$ is continuous across all three stationary states as drawn in Figure 2.

$H(x,h,G,\lambda) = m(wh) + bh - v(x) + \xi(G) + \lambda \cdot (\gamma x - \delta h)h$. $H_{xx} = -\alpha < 0$ (inner solution to $\max_x H(x,h,G,\lambda)$).
$H_h = \frac{\kappa}{h} + b + \lambda \cdot (\gamma x - 2\delta h)$, $H_{hh} = -\frac{\kappa}{h^2} - 2\lambda\delta < 0$ since $\lambda = \frac{v'(x)}{\gamma h} \geq 0$, $H_{hx} = \lambda\gamma$.
$H_{xx}H_{hh} - H_{hu}^2 = \alpha\left(\frac{\kappa}{h^2} + 2\lambda\delta\right) - \lambda^2\gamma^2$. Inserting the FOC $\lambda = \frac{v'(x)}{\gamma h}$ yields $H_{xx}H_{hh} - H_{hu}^2 = \alpha\left(\frac{\kappa}{h^2} + 2\frac{v'(x)}{\gamma h}\delta\right) - \left(\frac{v'(x)}{\gamma h}\right)^2\gamma^2 > 0$ if $\alpha\left(\kappa + 2v'(x)\frac{\delta}{\gamma}h\right) > v'(x)^2$. At $x = x_{\dot{h}=0}(h) = \frac{\delta}{\gamma}h$: $\alpha(\kappa + 2(a+\alpha x)x) > (a+\alpha x)^2$ or $\alpha\kappa + \alpha^2x^2 > a^2$. Equivalently (at $x = x_{\dot{h}=0}(h) = \frac{\delta}{\gamma}h$): $\alpha\kappa + \alpha^2\left(\frac{\delta}{\gamma}\right)^2 h^2 > a^2$ or $h^2 > \frac{a^2-\alpha\kappa}{\alpha^2\left(\frac{\delta}{\gamma}\right)^2}$. This is satisfied for all h if $\alpha > \frac{a^2}{\kappa}$.

Case 2: $a\rho > \gamma\kappa$ and $\frac{b}{\delta} > \frac{2\sqrt{\alpha\gamma(a\rho-\gamma\kappa)}+\alpha\rho+a\gamma}{\gamma^2}$. First, choose any a, ρ, γ, κ satisfying $a\rho > \gamma\kappa$. Second, choose $\alpha > \frac{a^2}{\kappa}$. Third, given a, ρ, γ, κ, α, choose $\overline{\left(\frac{b}{\delta}\right)} > \frac{2\sqrt{\alpha\gamma(a\rho-\gamma\kappa)}+\alpha\rho+a\gamma}{\gamma^2}$. Note that b (with $\delta = \overline{\left(\frac{b}{\delta}\right)} \cdot b$) or δ (with $b = \overline{\left(\frac{b}{\delta}\right)} \cdot \delta$) remain completely free.

Generating the three cases by a variation of a

Proposition 9 *If the economic motive for effort is not dominated by the economic motive ($b > \frac{\kappa\delta}{\rho} + \frac{\delta\alpha\rho}{\gamma^2}$: at $h = 0$, $x_{\dot{x}=0}(h)$ increases with a slope higher than $x_{\dot{h}=0}(h)$), then a variation of the simple cost of effort from a sufficiently low to a sufficiently large value generates all three cases: For small* **a** *all households converge to the same strictly positive level of human capital (see Figure 1a); for intermediate* **a** *the threshold case prevails (see Figure 2); for large* **a** *all households choose a path towards increasing passivity (see Figure 3). There are never multiple stationary solutions to the individual household problem if the non-economic motive for effort is too weak. Formally: If $b < \frac{\delta}{\gamma}\left(\frac{\alpha}{\gamma}\rho + a\right)$ then threshold dynamics occurs for no κ. If $b < \delta\left(\frac{\kappa}{\rho} + \frac{\rho\alpha}{\gamma^2}\right)$ then it occurs for no* **a**.

Proof. Consider Figures 1 to 3 of Section 3. If at $h = 0$ the slope of $\dot{x} = 0$-isocline is larger than the slope or the $\dot{h} = 0$-isocline (formally if $\frac{d}{dh}x_{\dot{x}=0}(0) = \frac{\gamma}{\alpha}\frac{(\rho b-\kappa\delta)}{\rho^2} > \frac{\delta}{\gamma} = \frac{d}{dh}x_{\dot{h}=0}(0)$ or if $b > \frac{\delta\rho\alpha}{\gamma^2} + \kappa\frac{\delta}{\rho}$), then a variation of the (simple) cost of effort a can generate all 3 cases. This is again best seen by considering the diagrams: In Figure 1a $\frac{d}{dh}x_{\dot{x}=0}(0) > \frac{d}{dh}x_{\dot{h}=0}(0)$. Starting from Figure 1a, raising a shifts down the function $x_{\dot{x}=0}(h)$, first generates Figure 2 and then, further raising a, Figure 3. More precisely, Case 1a occurs if $a \leq \gamma\kappa/\rho$, Case 2 occurs if $a > \gamma\kappa/\rho$, but not larger than the solution to $b = \frac{2\delta\sqrt{\alpha\gamma(a\rho-\gamma\kappa)}+\alpha\rho\delta+a\delta\gamma}{\gamma^2}$. Case 3a occurs if a is larger than this solution. It is obvious from the figure that if at $h = 0$ the slope of $\dot{x} = 0$-isocline is smaller than the slope or the $\dot{h} = 0$-isocline ($\frac{d}{dh}(x_{\dot{x}=0}(0)) < \frac{d}{dh}(x_{\dot{h}=0}(0))$ or $b < \frac{\kappa\delta}{\rho} + \frac{\delta\alpha\rho}{\gamma^2}$), then the threshold case (Case 2) occurs for no level of effort costs a. ■

8.2 Appendix: Simple Transfers

Apart from the stationary state at $h_* = 0$ the new dynamic system (9) has stationary states for $\frac{\gamma}{\alpha}\frac{\kappa\frac{1}{1+\frac{M}{wh}}+bh}{\rho+\delta h} - \frac{a}{\alpha} = \frac{\delta}{\gamma}h$ or for $Q(h,M) = \frac{\delta^2}{\gamma}h^3 - \left[B - \frac{\delta^2}{\gamma}\frac{M}{w}\right]h^2 - \left[C + B\frac{M}{w}\right]h + \rho\frac{a}{\alpha}\frac{M}{w} = 0$ with as before $C = \frac{\kappa\gamma-\rho a}{\alpha}$, $B = \left(\frac{b\gamma-a\delta}{\alpha} - \frac{\delta}{\gamma}\rho\right)$ and $D = \frac{\delta^2}{\gamma} > 0$. Note that for $M = 0$: $Q(h,0) = E(h,0) = \left(\frac{\delta^2}{\gamma}h^2 - Bh - C\right)h$. The polynomial function of degree 3 has up to three strictly positive real roots.

Lemma 10 *If $M > 0$, then $Q(h)$ has either no or two strictly positive roots.*

Proof. Since the coefficient $D = \frac{\delta^2}{\gamma}$ of the cubic term is positive $\lim_{h\to\infty} Q(h) = \infty$ and $\lim_{h\to-\infty} Q(h) = -\infty$. Furthermore $Q(0) = \rho\frac{a}{\alpha}\frac{M}{w} > 0$. From $\lim_{h\to-\infty} Q(h) = -\infty$ and $Q(0) > 0$ follows that one root must be negative. If $Q(\overline{h}) < 0$ for some $\overline{h} > 0$, then there must be a second root in the interval $(0,\overline{h})$. Since $\lim_{h\to\infty} Q(h) = \infty$ there must be a third root at some $h > \overline{h}$. ■

Lemma 11 *If $C > 0$ and $B < \sqrt{4\frac{\delta^2}{\gamma}\frac{a\rho}{\alpha}}$, then a small positive transfer turns Case 1 into the threshold Case 2, while a sufficiently large transfer turns Case 1 into Case 3. If $C > 0$ and $B > \sqrt{4\frac{\delta^2}{\gamma}\frac{a\rho}{\alpha}}$, then any transfer turns Case 1 into Case 2. The threshold human capital level h^{th} of Case 2 depends positively on the transfer and the strictly positive stable stationary h^* depends negatively on the transfer.*

Proof. $\lim_{h\to 0} x_{\dot{x}=0}(M,h) = \frac{\gamma}{\alpha}\frac{\kappa\frac{1}{1+\frac{M}{w\lim h}}+bh}{\rho+\delta h} - \frac{a}{\alpha} = -\frac{a}{\alpha}$ for all $M > 0$. At $M = h = 0$ the effect of raising M is dominated by the effect of the constant term, which raises $Q(0,M)$. Thus starting from Case 1 ($Q(h,0) = 0$ has one strictly positive root) $Q(h,M) = 0$ has two strictly positive solutions for small positive M (Case 2).

$\lim_{M\to\infty} Q(h,M)\frac{w}{M} = \frac{\delta^2}{\gamma}h^2 - Bh + \rho\frac{a}{\alpha} = 0$ has roots $h^{\infty}_{1,2} = \frac{B\pm\sqrt{B^2-4D\rho\frac{a}{\alpha}}}{2D}$. If $B > \sqrt{4D\rho\frac{a}{\alpha}}$ then both roots are positive. The household problem remains in the threshold Case 2 even for large M. If $B < \sqrt{4\frac{\delta^2}{\gamma}\frac{a\rho}{\alpha}}$ then there is no real root. The household problem is in Case 3 for sufficiently large M. ■

8.3 Appendix: Activating Welfare

8.3.1 Appendix: The utilitarian planner problem.

For later reference, I solve the more general **p-utilitarian problem** which generalizes the (equal-weight) (11) to

$$\max \int_{i\in I}\int_{t=0}^{\infty} e^{-\rho t}\left[p^i\left(m(c_t^i)+\xi G_t\right)+p^i(bh_t^i - v(x_t^i))\right]dtdi \tag{19}$$

$$\text{subject to } \int_{i\in I} c_t^i di + G_t = w\int_{i\in I} h_t^i di \text{ for all } t\geq 0$$

$$\dot{h}_t^i = (\gamma x_t^i - \delta h_t^i)h_t^i \text{ for all } t\geq 0,\ i\in I$$

for weights $p=\left(p^i\right)_{i\in I}\geq 0$, $\int_{i\in I} p^i di = 1$ and given $(h_0^i)_{i\in I}$. The equal-weight utilitarian problem is the special case with $p^i = 1$ for all $i\in I$.

Lemma 12 *Solving (19) is equivalent to solving a static and a dynamic problem:*
(a) ***Static planner problem****: Given $\{\widetilde{x}_t^i,\widetilde{h}_t^i\}_{i,t}$, the consumption allocation $\{\widetilde{c}_t^i,\widetilde{G}_t\}_{i,t}$ maximizes $\int_{i\in I}\left(p^i(m(c_t^i)+G_t\right)di$ subject to $\int_{i\in I} c_t^i di+\widetilde{G}_t = w\overline{h}_t$ in each period $t\geq 0$, where $\overline{h}_t=\int_{i\in I} h_t^i di$. The solution of the static planner problem*[19] *is*

$$\widetilde{c}_t^i = \begin{cases} p^i w\overline{h}_t \text{ if } \overline{h}_t < \frac{\kappa}{w\xi} \\ p^i\frac{\kappa}{\xi} \text{ if } \overline{h}_t\geq\frac{\kappa}{w\xi}\end{cases} \text{ and } \widetilde{G}_t = \begin{cases} 0 \text{ if } \overline{h}_t<\frac{\kappa}{w\xi} \\ w\overline{h}_t - \frac{\kappa}{\xi} \text{ if } \overline{h}_t\geq\frac{\kappa}{w\xi}\end{cases}. \tag{20}$$

(b) ***Dynamic planner problem****: $\{\widetilde{x}_t^i,\widetilde{h}_t^i\}_{i,t}$ maximizes $\int_{i\in I} p_x^i\int_{t=0}^{\infty} e^{-\rho t}\left[-v(x_t^i)+bh_t^i\right]dtdi$ subject to $\dot{h}_t^i=(\gamma x_t^i-\delta h_t^i)h_t^i$ for $i\in I, t\geq 0$, given h_0. The dynamic planner problem can be decomposed into independent optimization problems for each type i. If the economy is in the threshold case Section 3 (Proposition 3), then it remains so for any i and $p^i>0$. There exist $h^{th}(p^i)$ and $h^*(p^i)$ such that* $\lim_{t\to\infty} h_t = 0$ *for all $h_0 < h^{th}(p^i)$ and* $\lim_{t\to\infty} h_t = h^*(p^i)$ *for all $h_0 > h^{th}(p^i)$. See Figure 5a and b.*

The current value Hamiltonian of (19) is

$$H(\left(c_t^i,x_t^i,h_t^i,\lambda_t^i\right)_i,G_t,\mu_t) = \int_{i\in I} p^i m(c^i)di + b\int_{i\in I} p^i h^i di - \int_{i\in I} p^i v\left(x^i\right)di + \xi G\int_{i\in I} p^i di \tag{21}$$
$$+\int_{i\in I}\lambda^i\cdot(\gamma x^i-\delta h^i)h^i + \mu\left(w\int_{i\in I} h_t^i di - G - \int_{i\in I} c_t^i di\right)$$

FOC for c_t^i: $p^i m'(c_t^i)=\mu \Leftrightarrow p^i\frac{\kappa}{c_t^i}=\mu\Leftrightarrow c_t^i=\frac{p^i\kappa}{\mu}$ for all i,t. FOC for G: Either [$G>0$ and $\frac{\partial H}{\partial G}=0$] (interior solution) or [$G=0$ and $\frac{\partial H}{\partial G}<0$] (corner solution). In case of an interior solution $\frac{\partial H}{\partial G}=\xi-\mu=$

[19] Note that the set of static Pareto-optima can also be characterized by the extended Samuelson rule: $\int_{i\in I}\frac{u_G^i}{u_c^i}di\leq 1$ and $\left(\int_{i\in I}\frac{u_G^i}{u_c^i}di-1\right)\cdot G=0$

$0 \Leftrightarrow \xi = \mu$. With $c_t^i = \frac{p^i \kappa}{\xi}$ the resource constraint becomes $G_t = w\overline{h}_t - \int_{i \in I} c_t^i di = w\overline{h}_t - \frac{\kappa}{\mu} = w\overline{h}_t - \frac{\kappa}{\xi}$, which is strictly positive iff $\overline{h}_t > \frac{\kappa}{w\xi}$.

If $\overline{h}_t \leq \frac{\kappa}{w\xi}$ then $G_t = 0$ (corner solution) and $\int_{i \in I} c_t^i di = w\overline{h}_t \Leftrightarrow \frac{\kappa}{\mu} = w\overline{h}_t \Leftrightarrow \frac{\kappa}{w\overline{h}_t} = \mu \Rightarrow c_t^i = \frac{p^i \kappa}{\frac{\kappa}{w\overline{h}_t}} = p^i w\overline{h}_t$. The solution of the planner problem given any $\{x_t^i, h_t^i\}_{i,t}$ and corresponding $\overline{h}_t$, is therefore given by (20). This proves part (a) of the lemma.

FOC for x: $p^i v'\left(x^i\right) = \lambda^i \gamma h^i \Leftrightarrow \lambda^i = \frac{p^i v'(x^i)}{\gamma h^i} \Rightarrow \widehat{\lambda}^i = \widehat{v'(x^i)} - \widehat{h}^i$.

Adjoint equation for the costate variable of h:

$\dot{\lambda}^i = \rho\lambda^i - \frac{\partial H}{\partial h^i} = \rho\lambda^i - \left(bp^i + \lambda^i \cdot (\gamma x^i - \delta h^i) - \lambda^i \delta h^i + \mu w\right)$

$\widehat{\lambda}^i = \rho - \left(\frac{bp^i}{\lambda^i} + (\gamma x^i - \delta h^i) - \delta h^i + \frac{\mu w}{\lambda^i}\right) = \rho - \left(\frac{bp^i}{\lambda^i} + \widehat{h}^i - \delta h^i + \frac{\mu w}{\lambda^i}\right)$. Equalizing this with $\widehat{\lambda}^i$ from the FOC for x, using $\mu = \frac{\kappa}{w\overline{h}_t}$, $\lambda^i = \frac{p^i v'(x^i)}{\gamma h^i}$ and rearranging yields $\widehat{v_x(x^i)} = \rho - \left(\frac{\gamma\left(b + \frac{p^i}{p^i}\frac{\kappa}{c^i}w\right)}{v'(x^i)} - \delta\right) h^i$.

With $\widehat{v_x(x^i)} = \frac{v_{xx}}{v_x}\dot{x}$ this can be rewritten as

$$\dot{x} = \frac{v_x\left(\rho + \delta h^i\right) - \left(b + \frac{\kappa}{c^i}w\right)\gamma h^i}{v_{xx}} \tag{22}$$

Case (i): Low aggregate income, $\overline{h}_t \leq \frac{\kappa}{w\xi}$. In this case $c_t^i = p^i w\overline{h}_t$ and $\widehat{v'(x^i)} = \rho - \left(\frac{\gamma\left(b + \frac{1}{p^i}\frac{\kappa}{\overline{h}_t}\right)}{v'(x^i)} - \delta\right) h^i = 0$ iff $v'(x^i) = \frac{\gamma\left(b + \frac{1}{p_x^i}\frac{\kappa}{\overline{h}_t}\right)}{\frac{\rho}{h^i} + \delta} \Leftrightarrow a + \alpha x^i = \frac{\gamma\left(b + \frac{1}{p_x^i}\frac{\kappa}{\overline{h}_t}\right)h^i}{\rho + \delta h^i} \Leftrightarrow$

$$x^i_{\dot{x}=0}(h^i, \overline{h}_t \leq \frac{\kappa}{w\xi}) = \frac{\gamma\left(b + \frac{1}{p^i}\frac{\kappa}{\overline{h}_t}\right)h^i}{\alpha\left(\rho + \delta h^i\right)} - \frac{a}{\alpha}.$$

Thus in case (i), the phase-diagram for agent i depends on $\overline{h}_t$. The planner problem cannot be decomposed in independent individual problems. As we will see, conditions (28) and (29) guarantee that the economy starts and remains in Case (ii).

Case (ii): High aggregate income, $\overline{h}_t > \frac{\kappa}{w\xi}$. In this case $c_t^i = p^i\frac{\kappa}{\xi}$ and $\widehat{v'(x^i)} = \rho - \left(\frac{\gamma\left(b + \frac{w\xi}{p^i}\right)}{v'(x^i)} - \delta\right) h^i = 0$ iff $v'(x^i) = \frac{\gamma\left(b + \frac{w\xi}{p^i}\right)}{\frac{\rho}{h^i} + \delta} \Leftrightarrow a + \alpha x^i = \frac{\gamma\left(b + \frac{w\xi}{p^i}\right)h^i}{\rho + \delta h^i} \Leftrightarrow$

$$x^i_{\dot{x}=0}(h^i, \overline{h}_t > \frac{\kappa}{w\xi}) = \frac{\gamma\left(b + \frac{w\xi}{p^i}\right)h^i}{\alpha\left(\rho + \delta h^i\right)} - \frac{a}{\alpha}$$

For $\overline{h}_t > \frac{\kappa}{w\xi}$ and the specific case $p^i = 1$, the two isoclines are thus given by (12). With $A := \frac{\rho a}{\alpha}$, $B_p^i := \left(\frac{\gamma}{\alpha}\left(b + \frac{w\xi}{p^i}\right) - \delta\left(\frac{\rho}{\gamma} + \frac{a}{\alpha}\right)\right)$, and $D = \frac{\delta^2}{\gamma} > 0$ the (non-zero) stationary states are the solutions to

$$E_p^i(h) := -A + B_p^i h - Dh^2 = 0$$

$$h_{1,2}(p^i) = \frac{B_p^i \pm \sqrt{\left(B_p^i\right)^2 - 4DA}}{2D}. \tag{23}$$

Since $4DA > 0$, the solutions are real and strictly positive if $(B_p)^2 - 4DA > 0$ or if $b > \frac{\delta\left(2\sqrt{\alpha\gamma\rho a} + \delta(\alpha\rho + a\gamma)\right)}{\gamma^2} - \frac{w\xi}{p^i}$, which is satisfied for any $\xi \geq 0$ by (28). Furthermore, the utilitarian planner-economy starts in Case (ii) with $\overline{h}_t > \frac{\kappa}{w\xi}$ since $n^s h_0^s > \frac{\kappa}{w\xi}$ (Condition (29)) and remains in this case at $t = 0$ because $h_0^s > h_{n^s}^{\text{th}} := h_1(\frac{1}{n^s}) \geq h_1(p^i)$ (Condition (29) and $p^i \leq \frac{1}{n^s}$ in all cases that are relevant in the economy with two groups).[20]

Stability of the interior stationary solutions in Case (ii) The associated Jacobian of the system defined by $\dot{h}^i = (\gamma x^i - \delta h^i)h^i$ and (12) is $J = \begin{pmatrix} \frac{\partial \dot{h}}{\partial h} & \frac{\partial \dot{h}}{\partial x} \\ \frac{\partial \dot{x}}{\partial h} & \frac{\partial \dot{x}}{\partial x} \end{pmatrix} = \begin{pmatrix} \gamma x - 2\delta h & \gamma h \\ \frac{\delta v_x + \gamma\left(b + \frac{\xi}{p^i} w\right)}{v_{xx}} & (\rho + \delta h) - \dot{x}\frac{v_{xxx}}{v_{xx}} \end{pmatrix}$

$\dot{x} = \frac{\rho v_x - \left(\gamma\left(b + \frac{\xi}{p^i} w\right) - \delta v_x\right) h^i}{v_{xx}}$, $\frac{\partial \dot{x}}{\partial h} = \frac{\delta v_x + \gamma\left(b + \frac{\xi}{p^i} w\right)}{v_{xx}}$, $\frac{\partial \dot{x}}{\partial x} = \frac{d\frac{\rho v_x + \delta v_x h^i}{v_{xx}}}{dx} = \frac{\rho v_{xx} + \delta v_{xx} h^i}{v_{xx}} - \dot{x}\frac{v_{xxx}}{v_{xx}}$

At $\dot{h} = 0$ and $h \neq 0$: $J = \begin{pmatrix} -\delta h & \gamma h \\ \frac{(a + \alpha x)\delta - (b + \frac{\xi}{p^i} w)\gamma}{\alpha} & (\rho + \delta h) \end{pmatrix}$ and $\det J = -\delta h(\rho + \delta h) - \frac{(a + \alpha x)\delta - (b + \frac{\xi}{p^i} w)\gamma}{\alpha}\gamma h$.

At $x = \frac{\delta}{\gamma} h$, $\det J = -\delta h(\rho + \delta h) - \frac{(a + \alpha\frac{\delta}{\gamma} h)\delta - (b + \frac{\xi}{p} w)\gamma}{\alpha}\gamma h = \left[\frac{(b + \frac{\xi}{p^i} w)\gamma^2 - \delta(a\gamma + \alpha\rho)}{\alpha} - 2\delta^2 h\right] h$. With

$2\delta^2 h_{1,2}^i = 2\delta^2 \frac{B_p^i \pm \sqrt{\left(B_p^i\right)^2 - 4DA}}{2D} = \gamma\left(B_p^i \pm \sqrt{\left(B_p^i\right)^2 - 4DA}\right)$

$= \left(\frac{\left(\gamma^2\left(b + \frac{w\xi}{p^i}\right) - \delta(\alpha\rho + a\gamma)\right)}{\alpha} \pm \gamma\sqrt{\left(\frac{\gamma}{\alpha}\left(b + \frac{w\xi}{p^i}\right) - \delta\left(\frac{\rho}{\gamma} + \frac{a}{\alpha}\right)\right)^2 - 4\frac{\delta^2}{\gamma}\frac{\rho a}{\alpha}}\right)$,

$\det J(h_{1,2}) = -\left(\pm\gamma\sqrt{\left(\frac{\gamma}{\alpha}\left(b + \frac{w\xi}{p^i}\right) - \delta\left(\frac{\rho}{\gamma} + \frac{a}{\alpha}\right)\right)^2 - 4\frac{\delta^2}{\gamma}\frac{\rho a}{\alpha}}\right) h$

Thus $\det J(h_1) > 0$ and $\det J(h_2) < 0$. Therefore h_1 is unstable (whenever it exists, thus in Case 2) and h_2 is saddle-point stable (whenever it exists, thus in Cases 1 and 2).

Unstable Solution is a node without overlap As noted in Appendix 8.1 the unstable stationary solution $h_1(p^i)$ of (19) is a node and the policy function is continuous across all three stationary states if $h_1(p^i)$ lies in the concave domain of the Hamiltonian (Hartl et al. [2004]).

Lemma 13 *The Hamiltonian (21) of the planner problem is concave at the smaller inner solution of FOC iff* $h_1(p^i) \geq \frac{\gamma}{\delta}\frac{a}{\alpha}$.

Proof. With $H_{x^i} = -p^i(a + \alpha x^i) + \lambda^i \gamma h^i$, $H_{c^i} = p^i\frac{\kappa}{c^i} - \mu$, $H_{h_i} = p_x^i b + \lambda^i \cdot (\gamma x - 2\delta h) + \mu w$, $\mu = \frac{\kappa}{\overline{c}_t}$, one gets

[20] In the economy with two groups we have $n^u p^u + n^s p^s = 1$, $p^s, p^u \geq 0$ and therefore $p^s = \frac{1 - n^u p^u}{n^s} < \frac{1}{n^s}$ and $p^u = \frac{1 - n^s p^s}{n^u} < \frac{1}{n^u}$. Since the present paper only consider cases with $p^u \leq p^s$ it is sufficient to assume $h_0^s > h_1(\frac{1}{n^s})$ otherwise this would have to be strengthened to $h_0^s > \max h_1\{\frac{1}{n^s}, \frac{1}{n^u}\}$.

$$\mathcal{M} := \begin{pmatrix} H_{h_ih_i} & H_{h_ix_i} & H_{h^ic^i} \\ H_{x^ih^i} & H_{x^ix^i} & H_{x^ic^i} \\ H_{c^ih^i} & H_{c^ix^i} & H_{c^ic^i} \end{pmatrix} = \begin{pmatrix} -2\lambda^i\delta & \lambda^i\gamma & 0 \\ \lambda^i\gamma & -p^i\alpha & 0 \\ 0 & 0 & -p^i\frac{\kappa}{c^{i2}} \end{pmatrix}.$$

The three principle minors of $\mathcal{M}$ are $D_1 = -2\lambda^i\delta$, $D_2 = \det \begin{pmatrix} -2\lambda^i\delta & \lambda^i\gamma \\ \lambda^i\gamma & -p^i_x\alpha \end{pmatrix}$ and $D_3 = \det \mathcal{M}$. $H(x, h, G, \lambda)$ is concave if $D_1 \leq 0$ (satisfied), $D_2 \geq 0$ and $D_3 \leq 0$.
$D_2 = 2\lambda^i\delta p^i\alpha - \lambda^{i2}\gamma^2 < 0$ iff $2\delta p^i_x\alpha - \frac{p^iv'(x^i)}{\gamma h^i}\gamma^2 < 0$ iff $2\delta\alpha - \frac{v'(x^i)}{h^i}\gamma < 0$. At $x = x_{\dot{h}=0}(h^i) = \frac{\delta}{\gamma}h^i$, $D_2 \leq 0$ iff $2\delta\alpha - \frac{a+\alpha x^i}{\frac{\gamma x^i}{\delta}}\gamma \geq 0$ iff 0 iff $\alpha - \frac{a}{x^i} \geq 0$ iff $\begin{cases} x^i \geq \frac{a}{\alpha} \text{ for } x^i > 0 \\ x^i \leq \frac{a}{\alpha} \text{ for } x^i < 0 \end{cases}$, thus iff $x^i \geq \frac{a}{\alpha}$. $D_3 = -p^i\frac{\kappa}{c^{i2}}D_2 \leq 0$ iff $-p^i\frac{\kappa}{c^{i2}} \leq 0$ since $D_2 \geq 0$ (if $x^i \geq \frac{a}{\alpha}$). Thus, the Hamiltonian is concave at the smaller inner solution of FOC iff $x(h_1(p^i)) \geq \frac{a}{\alpha}$ or iff $h_1(p^i) \geq \frac{\gamma}{\delta}\frac{a}{\alpha}$. ■

For the equal-weight problem with $p^i = 1$, condition (28) guarantees the existence of a threshold (since $b + w\xi > b + n^s w\xi > \frac{\delta\left(2\sqrt{\alpha\gamma\rho a} + \delta(\alpha\rho + a\gamma)\right)}{\gamma^2}$). If in addition for $h_1(p^i = 1) > \frac{\gamma}{\delta}\frac{a}{\alpha}$, the Hamiltonian of the equal-weight problem is concave at $h_1(p^i = 1)$ and it follows that the unstable stationary solution $h_1(p^i = 1)$ of (21) is a node and the policy function is continuous across all three stationary states.[21] While the main text, in particular figures 5 and 6, implicitly make these assumptions,[22] Section 6 and Appendix **??** discuss planner-problems with noncontinuous policy functions.

Implementing the utilitarian solution (1) The Hamiltonian of (10) is $H(x, h, \lambda, G) = m[wh - M(h, \overline{h}^u, \overline{h}^s)] + bh - v(x) + \xi w\overline{h} + \lambda \cdot (\gamma x - \delta h)h$, where $\overline{h} = n^s\overline{h}^s + (1-n^s)\overline{h}^u = \int_{j\in I} h_t dj$. The general form of the equations of motion (4) and of the isoclines (7) are again unchanged. As in the baseline model, existence of an inner solution to $\max_x H$ requires $H_x = -v_x + \lambda\gamma h = 0$, which is sufficient since $H_{xx} = \alpha < 0$. At the maximum $v_x = \lambda\gamma h$. Taking the derivative with respect to time yields $v_{xx}\dot{x} = \dot{\lambda}\gamma h + \lambda\gamma\dot{h} = \dot{\lambda}\gamma h + v_x(\gamma x - \delta h)$. Inserting the adjoint equation $\dot{\lambda} = \rho\lambda - H_h = \rho\lambda - m_h - b - \lambda\cdot(\gamma x - 2\delta h)$ into this expression for $v_{xx}\dot{x}$ and using $v_x = \lambda\gamma h$ yields $v_{xx}\dot{x} = v_x(\rho + \delta h) - (m_h + b)\gamma h$. The general form of thy dynamic equation for x is the same as in the baseline model $\dot{x} = \frac{v_x(\rho+\delta h)-(m_h+b)\gamma h}{v_{xx}}$. With

[21] Note that these conditions are not mutually exclusive: The solution of the p-problem is in the threshold case for any $B_p \geq B^{\text{crit}} := 2\sqrt{AD}$. Since $h_1(p) = \frac{B_p - \sqrt{B_p^2 - 4AD}}{2D}$ is decreasing in B, it follows that $h_1(p) \leq h_1^{\text{crit}} = \frac{B^{\text{crit}}}{2D} = \frac{1}{\delta}\sqrt{\frac{\gamma\rho a}{\alpha}}$. $h_1^{\text{crit}} > \frac{\gamma}{\delta}\frac{a}{\alpha}$ iff $\alpha > \frac{a\gamma}{\rho}$. Thus, the two conditions (threshold condition $B_p > 2\sqrt{AD}$ and concavity condition $h_1(p) > \frac{\gamma}{\delta}\frac{a}{\alpha}$) are met for all parameter constellations with $\alpha > \frac{a\gamma}{\rho}$ and, given α, a, γ, ρ, with $B(p)$ larger than, but sufficiently close to B^{crit}.

[22] This simplifies the analysis, but is not essential for Theorem 7. It is essential that the threshold case prevails, which is guaranteed by (28). If the Hamiltonian is not concave at the instable solution $h_1(p = 1)$, the threshold $h^{th}(p = 1)$ may differ from $h_1(p = 1)$ (see Appendix **??**), in which case it should be clear in the proof of Theorem 7 that the relevant threshold $h^{th}(p = 1)$ to overcome may be larger than $h_1(p = 1)$.

$m_h = \frac{\kappa[w - M_h(h, \overline{h}_t^u, \overline{h}_t^s)]}{[wh - M(h, \overline{h}_t^u, \overline{h}_t^s)]}$ we now have

$$\dot{x} = \frac{v_x(\rho + \delta h) - \left(\frac{\kappa[w - M_h(h, \overline{h}_t^u, \overline{h}_t^s)]}{[wh - M(h, \overline{h}_t^u, \overline{h}_t^s)]} + b\right)\gamma h}{v_{xx}} \tag{24}$$

(2) Comparing (22) and (24), we see that the dynamic system with tax matches the dynamics of the solution to (19) if $\frac{\kappa}{c^i} w = \frac{\kappa[w - M_h]}{[wh - M]}$.

For $\overline{h}_t > \frac{\kappa}{w\xi}$ with utilitarian $\widetilde{c}^i(p) = p^i \frac{\kappa}{\xi}$, the tax-system should therefore, besides implementing this consumption allocation, satisfy $\frac{\kappa}{c^i} w = \frac{\kappa[w - M_h)]}{[wh - M]}$, thus $\frac{\kappa}{p^i \frac{\kappa}{\xi}} w = \frac{\xi}{p^i} w = \frac{\kappa[w - T_h)]}{[wh - T]}$ or $T_h = \frac{w\xi}{p^i \kappa}\left[\frac{\kappa}{\xi} - wh + M\right]$. Solving this differential equation yields

$$M^{k_i}(h_t^i, \overline{h}_t^u, \overline{h}_t^s) = wh_t^i + K_t^i \cdot e^{\frac{\xi w h_t^i}{p^i \kappa}}, \tag{25}$$

where K_t^i is a constant of integration, which may depend on $p^i, \overline{h}_t^u, \overline{h}_t^s$ (but of course not on h_t^i). The tax function (25) generates the same dynamic equations for (x^i, h^i) as the utilitarian solution for any K_t^i.

(3) To implement $\widetilde{z}(p)$, the system also has to yield at equilibrium $c_t^i = \widetilde{c}_t^i(p) = \frac{\kappa}{p^i \xi}$ for all i. Since at equilibrium $h_t^i = \overline{h}_t^{k_i}$, the budget constraint in (10) implies $c_t^i = w\overline{h}_t^{k_i} - M^{k_i}(h_t^i, \overline{h}_t^u, \overline{h}_t^s)$. Thus, we need $\frac{\kappa}{p^i \xi} = w\overline{h}_t^{k_i} - M^{k_i}(h_t^i, \overline{h}_t^u, \overline{h}_t^s)$ or, inserting (25), $\frac{\kappa}{p^i \xi} = -K_t^i \cdot e^{\frac{\xi w \overline{h}_t^{k_i}}{p^i \kappa}}$, which determines $K_t^i = -\frac{\kappa}{p^i \xi} \cdot e^{\frac{-\xi w \overline{h}_t^{k_i}}{p^i \kappa}}$. Inserting this into (25) yields $M(h_t^i, \overline{h}_t^u, \overline{h}_t^s) = wh_t^i - \frac{\kappa}{p^i \xi} \cdot e^{\frac{\xi w}{p^i \kappa}(h_t^i - \overline{h}_t^{k_i})}$ if $\overline{h}_t > \frac{\kappa}{w\xi}$. A similar reasoning applies to the case $\overline{h}_t \leq \frac{\kappa}{w\xi}$ requiring $c_t^i = \widetilde{c}_t^i(p) = p^i w \overline{h}_t$, so that we get:

$$M^{k_i}(h_t^i, \overline{h}_t^u, \overline{h}_t^s) = \begin{cases} wh_t^i - p^i w \overline{h}_t \cdot e^{\frac{h_t^i - \overline{h}_t^{k_i}}{p^i \overline{h}}} \text{ if } \overline{h}_t \leq \frac{\kappa}{w\xi} \\ wh_t^i - p^i \frac{\kappa}{\xi} \cdot e^{\frac{\xi w}{p^i \kappa}(h_t^i - \overline{h}_t^{k_i})} \text{ if } \overline{h}_t > \frac{\kappa}{w\xi}. \end{cases} \tag{26}$$

(4) *At* equilibrium with $h_t^i = \overline{h}_t^{k_i}$

$$M(h_t^i, \overline{h}_t^u, \overline{h}_t^s) = \begin{cases} wh_t^i - p^i w \overline{h}_t \text{ if } \overline{h}_t \leq \frac{\kappa}{w\xi} \\ wh_t^i - p^i \frac{\kappa}{\xi} \text{ if } \overline{h}_t > \frac{\kappa}{w\xi}. \end{cases} \tag{27}$$

Using $n^s p^s + (1 - n^s) p^u = 1$, the government budget constraint

$$G_t = \int_I M^{k_i}(h_t^i, \overline{h}_t^u, \overline{h}_t^s) di = \begin{cases} w\overline{h}_t - [n^s p^s + (1 - n^s) p^u] \, w\overline{h}_t = 0 \text{ if } \overline{h}_t \leq \frac{\kappa}{w\xi} \\ w\overline{h}_t - [n^s p^s + (1 - n^s) p^u] \, \frac{\kappa}{\xi} = w\overline{h}_t - \frac{\kappa}{\xi} \text{ if } \overline{h}_t > \frac{\kappa}{w\xi} \end{cases}$$

implies $G_t = \widetilde{G}_t$ so that the optimal path $\{G_t\}_{t \geq 0}$ is automatically implemented at equilibrium.

8.3.2 Appendix: The intra-group symmetric optimum

The intra-group symmetric solution Since at z', an initially skilled agent only internalizes the effect of her public good contribution on the n^s other agents of her own group s, the intra-group optimal consumption is larger and the incentive to exert effort weaker than at the (equal-weight) utilitarian solution. Furthermore,

Lemma 14 *The behavior of the agents of group s at their intra-group symmetric optimal solution without transfers ($G_t^u \geq 0$) is identical to their behavior at the utilitarian solution of Lemma 12 of the economy which replaces ξ by $n^s\xi$. In particular*

$$c_t'^s = \frac{\kappa}{n^s\xi} \text{ and } x'_{\dot{x}=0}(h_t) = \frac{\gamma}{\alpha}\frac{(n^s\xi w + b)h_t}{\rho + \delta h_t} - \frac{a}{\alpha} \text{ for } n^s h_t^s > \frac{\kappa}{w\xi} - \min\{0, G_t^u\}.$$

The corresponding threshold $h_{n^s}^{th}$ is larger than h_{ut}^{th} and the stable steady state $h_{n^s}^*$ smaller than h_{ut}^*.

Correspondingly, the threshold skill $h_{n^s}^{\text{th}}$ of the intra-group symmetric problem is larger than $h_{\text{ut}}^{\text{th}}$.
Proof. If $G_t^u \geq 0$ (no transfers), this follows as a corollary of Lemma 12 in for $p^u = 0$ and $p^s = 1/n^s$. Since the utility of agents of group s is linear in G_t^u and since $(h_t^u)_{t\geq 0}$ is exogenously given for these agents, $(h_t^u)_{t\geq 0}$ does not affect the s-optimal intra-group effort and skill $(x_t'^s, h_t'^s)_{t\geq 0}$. More generally (for unconstrained G_t^u) an additional corner solution arises when gross income wh_t^s is smaller than the transfer $-\frac{G_t^s}{n^s w}$:

$$c_t'^s = \begin{cases} 0 \text{ if } h_t^s < -\frac{G_t^l}{n^s w} \\ wh_t^s + \min\{0, \frac{G_t^u}{n^s}\} \text{ if } -\frac{G_t^u}{n^s w} \leq h_t^s < \frac{\kappa}{n^s w\xi} - \min\{0, \frac{G_t^u}{n^s w}\} \\ \frac{\kappa}{\xi} \text{ if } h_t^s \geq \frac{\kappa}{n^s w\xi} - \min\{0, \frac{G_t^u}{n^s w}\} \end{cases}$$

Since the conditions guaranteeing that the economy starts and remains in the threshold case are stronger here than for the equal-weight planner problem, I quantify the condition in terms of the intra-group problem: ξ and h_0^s are sufficiently large if

$$b + n^s w\xi > \frac{2\delta\sqrt{\alpha\gamma a\rho} + \alpha\rho\delta + a\delta\gamma}{\gamma^2} \tag{28}$$

$$h_0^s > \max\left\{h_{n^s}^{\text{th}}, \frac{\kappa}{n^s w\xi}\right\} \tag{29}$$

where $h_{n^s}^{\text{th}} = \frac{B_{n^s} - \sqrt{(B_{n^s})^2 - 4DA}}{2D}$, with $A := \frac{\rho a}{\alpha}$, $B_{n^s} := \left(\frac{\gamma}{\alpha}(b + n^s w\xi) - \delta\left(\frac{\rho}{\gamma} + \frac{a}{\alpha}\right)\right)$, and $D = \frac{\delta^2}{\gamma} > 0$. (28) is equivalent to $B_{n^s} > 2\sqrt{AD}$ and guarantees the existence of a strictly positive inner stationary solution. Note that if the conditions of the threshold case in the baseline model are satisfied, (28) holds for any $\xi \geq 0$. $n^s h_0^s > \frac{\kappa}{w\xi}$ is sufficient to guarantee that the s initially want to provide the public good and that $n^s h_0^s > h_{n^s}^{\text{th}}$ is sufficient to guarantee that this remains true for later periods (provided that the s don't have to pay too large transfers to the u, e.g. provided $G_t^u > -(n^s h_0^s - \frac{\kappa}{w\xi})$).

For $h_t^s \geq \frac{\kappa}{n^s w\xi} - \min\{0, \frac{G_t^u}{n^s w}\}$ there is strictly positive public good provision ($G_t = G_t^s + G_t^u > 0$) and $c_t'^s = \frac{\kappa}{\xi}$. For $G_t^u \geq 0$ or for sufficiently small $|G_t^u|$ the condition $h_t^s > \frac{\kappa}{n^s w\xi}$ is sufficient to guarantee strictly positive public good provision and $c_t'^s = \frac{\kappa}{\xi}$. ■

Implementing the intra-group symmetric first best given transfers For $t \in (0,T)$, the s should be led to play their symmetric best solution. In the case without transfers ($G_t^u \geq 0$), repeating the steps of the previous paragraph but starting by equalizing $w\xi$ by $n^s w\xi$ and $\widetilde{c}_t^i = w\overline{h}_t$ by $\widetilde{c}_t^i = w\overline{h}_t^s$ leads to

$$M_{\text{sym}}^{k_i}(h_t^i, \overline{h}_t^u, \overline{h}_t^s) = \begin{cases} wh_t^i - w\overline{h}_t^s \cdot e^{\frac{h_t^i - \overline{h}_t^s}{\overline{h}_t^s}} & \text{if } n^s\overline{h}_t^s \leq \frac{\kappa}{w\xi} \\ wh_t^i - \frac{\kappa}{n^s\xi} \cdot e^{\frac{\xi w}{n^{k_\kappa}}(h_t^i - \overline{h}_t^s)} & \text{if } n^s\overline{h}_t^s > \frac{\kappa}{w\xi}. \end{cases} \tag{30}$$

With transfers from s to u ($G_t^u < 0$) and sufficiently large h_t^s ($h_t^s \geq \frac{\kappa}{n^s w\xi} - \min\{0, \frac{G_t^u}{n^s w}\}$), we have seen that it remains optimal for the s to consume $c_t'^s = \frac{\kappa}{\xi}$ and to spend the remainder of their post-transfer income on the public good. As long as $h_t^s \geq \frac{\kappa}{n^s w\xi} - \min\{0, \frac{G_t^u}{n^s w}\}$, (intra-group) optimal tax (contribution to transfer and to public good provision) for the s does not depend on G_t^u, so that that the same is true for the tax-function implementing optimal behavior.

8.3.3 Activation: Proof of Theorem 7

Ad Step 2a: Activation. Fix any $\widetilde{h} > h_{\text{ut}}^{\text{th}}$. For any $h_t^u > \widetilde{h}$ the policy is fixed by the utilitarian solution $x^{\text{ut}}(h)$. (In what follows, $\widetilde{h}$ will be chosen sufficiently large to make sure that the utilitarian solution Pareto dominates z' once $h_t^u > \widetilde{h}$). Activation is achieved by imposing a sufficiently strong effort $\widetilde{x}_t^u$:

$$\underline{x} := \frac{\Delta + \delta\widetilde{h}}{\gamma} \tag{31}$$

for $h < h_{\text{ut}}^{\text{th}}$ and for some $\Delta > \rho$. Also note that the optimal effort $\widetilde{x}_t^u(h)$ is bounded from above by $\overline{x} := x_{\dot{x}=0}^u(0) := \frac{\gamma}{\alpha}\frac{\eta\kappa}{\rho} - \frac{a}{\alpha}$.

The lower bound on $\widetilde{x}_t^u$ makes sure that $\widetilde{h}_t^u$ grows sufficiently fast: With $\widehat{\widetilde{h}}_t^u = \gamma\widetilde{x}_t^u - \delta h_t^u > \gamma\widetilde{x}_t^u - \delta\widetilde{h}$ for $h_t^u < h_{\text{ut}}^{\text{th}} < \widetilde{h}$ condition (31) implies

$$\widehat{\widetilde{h}}_t^u > \Delta > \rho \text{ for } h_t^u < h_{\text{ut}}^{\text{th}}. \tag{32}$$

Similarly, since $\widehat{\widetilde{h}}_t^u = \gamma\widetilde{x}_t^u - \delta h_t^u < \gamma\widetilde{x}_t^u$ for $h_t^u > 0$, the upper bound on x implies

$$\widehat{\widetilde{h}}_t^u < \gamma\overline{x} < \infty \text{ for } h_t^u \leq \widetilde{h}. \tag{33}$$

(32) and (33) make sure that h_t^u reaches $h_{\text{ut}}^{\text{th}}$ at strictly positive finite times, say at T_1 with $0 < T_1 < T$. For $h_t^u < h_{\text{ut}}^{\text{th}}$ choose any $\widetilde{x}_t^u$ satisfying

$$\underline{x} := \frac{\Delta + \delta\widetilde{h}}{\gamma} \leq \widetilde{x}_t^u < \overline{x} \tag{34}$$

Ad Step 3b: $\widetilde{z} \succ_s z'^s$. Since $G_t'^u \leq wh_t^u - c_t^u \leq wh_0^u - c_t^u$, for sufficiently small h_0^u condition (18) is equivalent to

$$\begin{aligned}
\int_0^T e^{-\rho t}\widetilde{G}_t^u dt &> 0 \\
\int_0^T e^{-\rho t}\left(w\widetilde{h}_t^u - \widetilde{c}^u\right)dt &> 0 \\
\int_0^T e^{-\rho t} w\widetilde{h}_t^u dt &> \widetilde{c}^u \frac{1-e^{-\rho T}}{\rho}
\end{aligned}$$

Equation (32) makes sure that $\widetilde{h}_t^u$ rises at a rate larger than Δ till time T_1. A sufficient condition for (18) is

$$\begin{aligned}
\int_0^{T_1} e^{-\rho t} w e^{\Delta t} h_0^u dt &\geq \widetilde{c}^u \frac{1-e^{-\rho T_1}}{\rho} \\
wh_0^u \frac{e^{(\Delta-\rho)T_1}-1}{\Delta-\rho} &\geq \widetilde{c}^u \frac{1-e^{-\rho T_1}}{\rho} \\
\widetilde{c}^u &\leq \frac{e^{(\Delta-\rho)T_1}-1}{\Delta-\rho}\frac{\rho}{1-e^{-\rho T_1}} wh_0^u.
\end{aligned}$$

The left hand side of this inequality is strictly positive iff $\Delta > \rho$, which is satisfied if η is sufficiently large given $\widetilde{h}$. Assuming a sufficiently large η, approval by the initially skilled ($\widetilde{z} \succ_s z'$) is therefore guaranteed if we set

$$\bar{c}^u := \frac{e^{(\Delta-\rho)T_1}-1}{\Delta-\rho}\frac{\rho}{1-e^{-\rho T_1}} wh_0^u. \tag{35}$$

Note that $\int_0^{T_1} e^{-\rho t}\widetilde{G}_t^u dt > 0$ implies $w\widetilde{h}_t^u > \widetilde{c}^u$ for all $t > T_1$ (since h_t^u is increases over time) such that (35) is sufficient for $\int_0^T e^{-\rho t}\widetilde{G}_t^u dt > 0$.

Ad Step 4: Approval by the beneficiary. Using (17) and (35), $\ln(\widetilde{c}_t^u) - \ln(c_t^u) = \ln \frac{\widetilde{c}^u}{c_t^u} \geq \ln \frac{\bar{c}^u}{wh_0^u} = \ln \frac{e^{(\Delta-\rho)T_1}-1}{\Delta-\rho}\frac{\rho}{1-e^{-\rho T_1}}$. By (32) $\Delta - \rho > 0$ and by (33) and $\lim_{h_0^u \to 0} T_1 = \infty$. Therefore $\lim_{h_0^u \to 0} \frac{\widetilde{c}^u}{c_t^u} = \infty$ and $\lim_{h_0^u \to 0} \frac{\ln \widetilde{c}^u}{\ln c_t^u} = \infty$.