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A LIFE-CYCLE MODEL WITH AMBIGUOUS SURVIVAL BELIEFS

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Abstract

On average, "young" people underestimate whereas "old" people overestimate their chances to survive into the future. We adopt a Bayesian learning model of ambiguous survival beliefs which replicates these patterns. The model is embedded within a non-expected utility model of life-cycle consumption and saving. Our analysis shows that agents with ambiguous survival beliefs (i) save less than originally planned, (ii) exhibit undersaving at younger ages, and (iii) hold larger amounts of assets in old age than their rational expectations counterparts who correctly assess their survival probabilities. Our ambiguity-driven model therefore simultaneously accounts for three important empirical findings on household saving behavior.

JEL Classification: D91, D83, E21.

Keywords: Cumulative prospect theory; Choquet expected utility; Dynamic inconsistency; Life-cycle hypothesis; Saving puzzles

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1 Introduction

One important element of life-cycle models of consumption and saving is the process of how individuals form and revise beliefs about their life expectancy when they grow older. In line with Muth's (1961) rational expectations paradigm it is common in the literature to consider expected utility maximizing agents whose updated subjective beliefs coincide with objective conditional survival probabilities. Only recently, researchers have focused on subjective assessments of survival probabilities which deviate from projected life-table survival rates on the aggregate level.[1] According to the Health and Retirement Study (HRS), on average, younger people strongly underestimate their (relatively high) probability to survive to some target age. At the same time older people strongly overestimate their lower survival probability. Such patterns can neither be reconciled with the rational expectations paradigm nor with models of rational Bayesian learning according to which subjective beliefs converge to objective probabilities when people gain more experience, i.e., grow older.

In addition, recent empirical findings on household saving behavior proved to be puzzling for the standard "workhorse"-life-cycle model à la Modigliani and Brumberg (1954) and Ando and Modigliani (1963) which assumes rational agents with perfect foresight throughout their life. For example, Laibson et al. (1998) and Bernheim and Rangel (2007) report large gaps between self-reported behavior and self-reported plans and/or preferences. Generally, people save less for retirement than actually planned (Choi et al. 2006). Such phenomena of dynamic inconsistency have been analyzed within models of *hyperbolic time-discounting* and bounded self-control. Building on the early work by Strotz (1955) and Pollak (1968), Laibson et al. (1998) find that exponential consumers save more than hyperbolic consumers. Besides these tendencies for undersaving and dynamically inconsistent behavior a well-known puzzle within the standard life-cycle framework is that people hold large amounts of assets still late in life and dissave less at the end of their life than predicted by the standard model (see, e.g., De Nardi et al. 2010; Hurd and Rohwedder 2010; Lockwood 2013).

This paper investigates in how far the aforementioned mistakes about assessing the prospect of survival may explain these empirical findings on saving behavior. To this purpose we merge a model of subjective survival belief formation with an otherwise standard life-cycle model. To model biases in survival beliefs as reported in the HRS, we adopt a simplified version of the Bayesian learning model under ambiguity developed by Ludwig and Zimper (2013). Agents of this learning model are decision makers whose

[1] For early contributions on the difference between subjective survival beliefs and objective survival probabilities see, e.g., Manski (2004), McFadden et al. (2005) and references therein.

preferences can be represented by *Choquet expected utility* (CEU) *theory* (Schmeidler 1989; Gilboa 1987) or, equivalently, by *cumulative prospect theory* (CPT) (Tversky and Kahneman 1992; Wakker and Tversky 1993).[2] More specifically, we model ambiguous survival beliefs as non-additive probabilities in the sense of Chateauneuf et al. (2007) which take individuals' psychological attitudes into account; notably ambiguity with respect to objective survival probabilities and a degree of relative optimism with which this ambiguity is resolved. Because we derive updated survival beliefs from a model of Bayesian learning under ambiguity, our model goes beyond a mere static application of CPT or CEU theory to survival beliefs as, e.g., in Bleichrodt and Eeckhoudt (2006) and in Halevy (2008). In contrast to these ad hoc CPT models, which fix a unique probability weighting function, our axiomatic approach towards Bayesian updating of CEU preferences[3] gives rise to a sequence of age-dependent probability weighting functions when we transform objective survival probabilities into subjective beliefs.

Based on our dynamic model of ambiguous survival beliefs, we use an otherwise standard stochastic life-cycle consumption model to compare the consumption and saving behavior of CEU agents with rational expectations (RE) agents who are nested within our approach as a special case. Whenever CEU agents do not reduce to RE agents, our life-cycle maximization problem gives rise to dynamically inconsistent behavior. We restrict attention to 'naive' agents. In contrast to a 'sophisticated' agent, who is fully aware of her dynamically inconsistent behavior, a naive agent does not anticipate that her future-selves have strict incentives to deviate from her ex ante optimal consumption plan. To consider naive rather than sophisticated agents is in line with empirical evidence that supports the relevance of the former (cf. O'Donoghue and Rabin (1999) and the literature cited therein).

Qualitative analysis for a simple three-period life-cycle model—relegated to the supplementary Appendix C—shows that naive CEU agents exhibit undersaving behavior relative to their RE counterparts if they sufficiently underestimate objective survival probabilities at young ages. Furthermore, at older ages they need to moderately overestimate their survival chances in order to save less than originally planned. Finally, CEU agents save more out of cash on hand in the intermediate model period than the corresponding RE agent. However, whether asset holdings in the final period are higher for the CEU agent depends on the interplay between underestimation at younger ages and overestimation at older ages. Whether these conditions hold and how relevant the biases in beliefs are for generating saving puzzles are quantitative questions.

[2] Restricted to outcomes which do not include losses but only gains (as in our model), CPT is identical to CEU theory.

[3] See, e.g., Gilboa and Schmeidler (1993), Eichberger et al. (2007), Zimper and Ludwig (2009) and Zimper (2012) for related work on the decision theoretic foundations.

To address these quantitative questions we calibrate the stochastic quantitative life-cycle model to the data. At this stage, the strength of our structural survival beliefs model comes into play because it allows us to characterize the entire survival beliefs distribution while there are only specific data points available from the HRS. We find that underestimation of survival probabilities at younger ages leads to undersaving for retirement and that dynamic inconsistency implies lower savings than originally planned. Moreover, due to the overestimation of survival probabilities at older ages, CEU agents decumulate assets at lower rates and eventually exhibit higher asset holdings than their RE counterparts. Our quantitative findings can be compressed in the following numbers: CEU agents with ambiguous survival beliefs at working age have a saving rate of 21.9% on average compared to a rational expectations model with an average saving rate of 22.8%. The realized saving rate is 2.8 percentage points lower than what the CEU agent at age 20 actually planned to save. Our model predicts average asset holdings at age 85 (95) of 46.4% (23.5%) of the assets at age 65 which is 11.5 (15.8) percentage points higher than respective values for rational agents.

These findings support our modeling approach. However, there still exists a gap between model-generated and empirical saving rates and asset holdings (cf. our discussion in Sections 5.3- 5.4). Of course, this does not come as a surprise because it is implausible that a stylized model of ambiguous survival beliefs alone can fully explain people's life-cycle decisions.

Finally, observe that the standard explanation for time inconsistency and undersaving at young ages in the form of *hyperbolic time discounting* models (Laibson et al. 1998; Angeletos et al. 2001) cannot account for high old-age asset holdings. Similar, the standard explanations for insufficient asset decumulation at old ages in the form of *bequest* (Hurd 1989; Lockwood 2013) and *precautionary savings motives* (Palumbo 1999; De Nardi et al. 2010) alone cannot explain undersaving at young ages. Our model of ambiguous survival beliefs adds to these existing explanations whereby it simultaneously generates all three stylized findings; namely, (i) time inconsistency of agents, (ii) undersaving at younger ages and (iii) high asset holdings at old age.

The remainder of our paper is organized as follows. Section 2 motivates and presents our first building block, a parsimonious model of ambiguous survival beliefs. In Section 3 we combine this model with a quantitative multi-period stochastic life-cycle model. Calibration is outlined in Section 4 and results of the quantitative analysis are presented in Section 5. Finally, Section 6 concludes our analysis. Appendix A recalls formal definitions from Choquet decision theory. Appendix B sketches the construction of ambiguous survival beliefs through a model of Bayesian learning under ambiguity by Ludwig and Zimper (2013). Supplementary Appendix C contains the analytical three-period model.

2 Ambiguous Survival Beliefs

2.1 Biases in Survival Beliefs

As point of departure, consider the subjective survival beliefs elicited in the Health and Retirement study (HRS). Respondents are asked about their assessment of the probability to survive from some interview age up to a specific target age. Target age is mostly 10 to 15 years in advance, see Table 1.

Table 1: Interview and Target Age

Age at Interview	Target Age
≤ 69	80
$70 - 74$	85
$75 - 79$	90
$80 - 84$	95
$85 - 89$	100

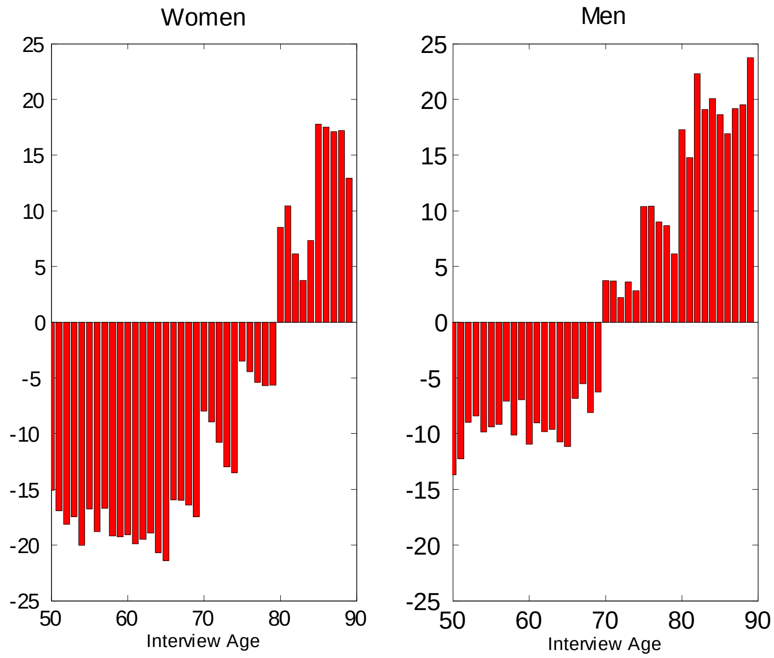
Source: RAND HRS Data Documentation.

Figure 1 shows aggregated data from the HRS by plotting average age-specific biases in survival beliefs—the difference between the respective average subjective belief and the average objective data—for three waves of the HRS between 2000 and 2004.[4] We observe that relatively "young"—younger than age 65-70—respondents underestimate whereas relatively "old"—above age 70—respondents overestimate their chances to survive into the future. For example, an average 65 year old women underestimates her objective probability to become 80 years by about 20 percentage points. Respondents between ages 85 and 89 in the sample exhibit an average overestimation by about 15 to 20 percentage points.

This age-specific pattern of subjective survival beliefs is a well-established stylized fact and has been confirmed by various other studies using different data sets. Hammermesh (1985) found that subjective survival rate functions are generally flatter than their objective counterpart implying underestimation at younger ages and overestimation at older ages. Similar findings have been described by Elder (2013) for the US and by Peracchi and Perotti (2010) for European countries using the Survey of Health, Ageing

[4] Objective data are based on cohort life-tables so that future trends in life expectancy are appropriately taken into account. Ludwig and Zimper (2013) also show that neither cohort effects nor selectivity issues are a concern. Furthermore, these patterns are robust to focal point answers at probabilities of $0\%, 50\%$ and 100%.

Figure 1: Relative difference of subjective survival probabilities and cohort data



Notes: This graph shows deviations in percentage points of subjective survival probabilities from objective data. Objective survival rates are based on cohort life table data. Future objective data is predicted with the Lee-Carter procedure (Lee and Carter 1992). Each bar depicts the difference of unconditional probabilities to survive to a specific target age, cf. Table 1.

Source: Own calculations based on HRS, Human Mortality Database and Social Security Administration data.

and Retirement in Europe (SHARE). Wu et al. (2013) highlight a related fact in aggregate data using the 2011 Australian "Retirement Plans and Retirement Incomes: Pilot Survey" which has a richer set of survival questions than the HRS. These data indicate that people of a certain (interview) age underestimate probabilities in the near future whereas they overestimate survival rates for the distant future.

The biases of subjective survival perceptions from objective life-table data shown in Figure 1 are expected to have significant implications for household's consumption and saving decisions. A number of recent studies confirms this. For example, Salm (2010) estimates that a 1 percent increase in the subjective probability of mortality reduces annual future consumption of non-durable goods by around 1.8 percent. Bloom et al. (2006) find that an increased subjective survival probability leads to higher wealth accumulation thereby confirming results of Hurd et al. (1998).

Incorporating subjective survival beliefs in structural dynamic household models however requires knowledge of the entire probability distribution while, in general, there are only few data points available. In the HRS, for example, only specific unconditional average subjective survival probability for each interview age is observed. A number of recent studies therefore estimate subjective survival beliefs functions by assuming specific hazard functions in order to adjust the aggregate differences of subjective data with the objective counterpart, either by a constant "subjective scaling factor" (e.g., Gan et al. 2005), by assuming distributions of subjective scaling factors (e.g., Bissonnette et al. 2011; Khwaja et al. 2007), or by allowing the adjustment factor to vary with target age (e.g., Wu et al. 2013). We add to this literature by using an ambiguity-driven model which has an axiomatic decision theoretic foundation and provides a structural framework to interpret as well as to inter- and extrapolate the data. The next Subsection describes this approach.

2.2 Parsimonious Model of Ambiguous Survival Beliefs

The biases in Figure 1 cannot be accommodated by a standard Bayesian learning model where the agent learns more relevant (statistical) information about her future survival chances as she grows older. Such a standard model would imply convergence to the objective probability.[5] To 'explain' the biases in Figure 1, we follow Ludwig and Zimper (2013) who set up a closed-form model of Bayesian learning under ambiguity within the framework of CEU theory which gives rise to a parsimonious notion of *ambiguous survival beliefs*.

Technically speaking, this model is based on Choquet decision theory such that survival beliefs are modeled as conditional *neo-additive capacities* (Chateauneuf et al. 2007) which are updated in accordance with the *Generalized Bayesian update rule* (Pires 2002; Eichberger et al. 2007). Neo-additive capacities are used in the literature[6] to approximate inversely S-shaped probability weighting functions typically elicited for CPT (cf., e.g., Tversky and Kahneman 1992; Wu and Gonzalez 1996; 1999). More importantly, in contrast to additive probabilities, the conditional neo-additive survival beliefs constructed in this paper can replicate the patterns of Figure 1 because they do not necessarily converge through Bayesian updating to the objective probabilities. For the reader's benefit we present a mathematically rigorous review of Choquet decision theory with neo-additive capacities in Appendix A and of the Ludwig and Zimper (2013) learning model in Appendix B. In what follows, we only restate the learning model's parsimonious characterization of ambiguous survival beliefs.

[5] This is implied by consistency results for Bayesian estimators. The seminal contribution is Doob (1949).

[6] See, e.g., Wakker (2010), Abdellaoui et al. (2011), and Ludwig and Zimper (2013).

Fix some $T \geq h \geq 0$ with the interpretation that the agent perceives it as possible to live until the end of period T whereas she perceives it as impossible to live longer than T. Denote by $\delta \in [0,1]$ an initial degree of ambiguity and by $\lambda \in [0,1]$ a psychological bias parameter which measures whether the agent resolves her ambiguity through over- or rather through under-estimation of the true probability. The following result is based on Ludwig and Zimper (2013) and derived in some detail in Appendix B[7]:

Proposition 1. *Denote the objective probability to survive from k to t by $\psi_{k,t}$ and fix age-independent parameters $\delta, \lambda \in [0,1]$. The h-old agent's age-dependent ambiguous belief to survive from age k to target age $t, \nu_{k,t}^{h}$, is given by*

$$\nu_{k,t}^{h} = \delta_h \cdot \lambda + (1 - \delta_h) \cdot \psi_{k,t}$$

such that

$$\delta_h = \frac{\delta}{\delta + (1-\delta) \cdot \frac{1}{1+\sqrt{h}}}$$

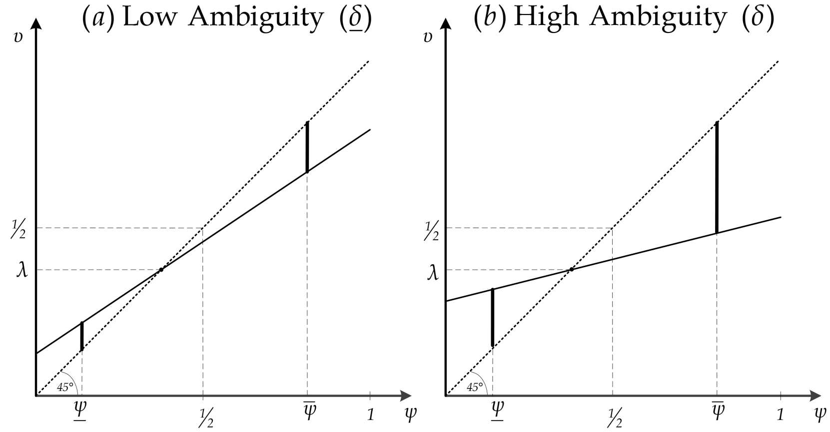
for $\psi_{k,t} \in (0,1)$ and $\nu_{k,t}^{h} = \psi_{k,t}$ for $\psi_{k,t} \in \{0,1\}$.

For all $\psi_{k,t} \in (0,1)$, the h-old agent's belief to survive from age k to some target age t is thus formally described as an age-dependent weighted average of the objective survival probability with weight $1 - \delta_h$ and the psychological bias parameter λ with weight δ_h. In the absence of ambiguity, i.e., $\delta = 0$, we have for all h that $\nu_{k,t}^{h} = \psi_{k,t}$ so that all ambiguous survival beliefs reduce to objective survival probabilities and the standard rational expectations approach is nested as a special case. For any positive ambiguity, i.e., $\delta > 0$, however, the dynamics of the model imply that agents exhibit more pronounced ambiguity attitudes with increasing age. A stylized representation of the model's dynamics is contained in Figure 2. Interior objective survival rates $\psi^h \in (0,1)$ are mapped into corresponding subjective survival rates ν^h by a linear transform which is (i) flatter than the 45-degree line and (ii) becomes flatter with increasing age h. Furthermore, the ν^h-line intersects with the 45-degree line at the relative optimism parameter λ. Note, that both the underestimation and the overestimation of survival probabilities become more pronounced with higher ambiguity. Increasing ambiguity with

[7] In fact, the model used here is a simplified version of the Ludwig and Zimper (2013) model, which merges a standard rational Bayesian learning (RBL) model (with some initial biases in prior beliefs in the additive part of the model) with the model of ambiguous survival beliefs. For sake of simplicity, we here ignore the initial bias in the RBL part so that any bias between objective survival probabilities and subjective beliefs is exclusively ambiguity-driven.

age inherent in our model together with lower objective survival rates for older agents (lower than λ which is consistent with our empirical estimates, cf. Subection 4.3) implies an increasing importance of overestimation of survival probabilities.

Figure 2: Dynamics of the Subjective Survival Belief Model



Notes: This graph depicts the subjective survival belief ν compared to the objective counterpart ψ. Panel (a) shows the case for low ambiguity $\underline{\delta}$ (young agent). The deviation from the 45-degree line is only modest implying that both underestimation of high $\bar{\psi}$ and overestimation of low $\underline{\psi}$ objective probabilities are small. Panel (b) shows high ambiguity $\bar{\delta}$ (older agent). The subjective survival line is more horizontal implying that under- and overestimation is more pronounced.

Bleichrodt and Eeckhoudt (2006) and Halevy (2008) also apply CPT probability weighting functions to survival beliefs. Because these authors use constant (i.e., age-independent) probability weighting functions, their decision theoretic approach falls under *rank dependent utility theory* (RDU) (Quiggin 1981; 1982) according to which the additive probability measure (and its Bayesian updates) of a probabilistically sophisticated decision maker is transformed by a fixed weighting function.

The decision theoretic foundation of ambiguous survival beliefs by Proposition 1 goes beyond RDU theory (and beyond probabilistic sophistication) because updating under ambiguity implies the violation of Savage's (1954) sure-thing principle in a Bayesian learning context.[8] More precisely, we construct in Appendix 2.3 age-dependent σ-algebras $\mathcal{F}^h$, $h = 1, ..., T$, which can be interpreted as different *sources of Ellsberg-like uncertainties* in the sense of Wakker (2010) and Abdellaoui et al. (2011). Under this interpretation, an agent of fixed age h can be described as an RDU (i.e., a probabilistically sophisticated) decision maker such that the additive probability $\psi_{k,t}$ has been transformed into the neo-additive probability $\nu^h_{k,t}$ by an age-specific probability weighting function. However, across different ages, i.e., for different sources of Ellsberg-like

[8]For technical details see the Appendix and references therein.

uncertainty, our decision maker is no longer probabilistically sophisticated so that her preferences can no longer be descried by a given probability weighting function as in RDU theory but rather by different probability weighting functions corresponding to different ages.

2.3 Heuristic Interpretations of Age-increasing δ_h

The Bayesian learning model underlying Proposition 1 assumes that the flow of statistical survival information is a standard filtration process so that the representative agent receives more information when she grows older. In this environment of Bayesian learning under ambiguity, more information coincides with "ex ante less likely", i.e., "surprising", information to the effect that an ambiguity prone decision maker expresses even more ambiguity in the face of more statistical information. As a formal consequence, ambiguity in survival beliefs, as expressed by the δ_h parameter, increases with age.

Although the age-increasing δ_h is thus a rather mechanical consequence of the underlying decision-theoretic assumptions, this formal feature captures the intuitive notion that, as the objective risk of survival becomes less likely, agents attach less and less weight to this objective probability. According to our estimates of δ and λ, presented in Section 4, objective survival probabilities $\psi_{k,t}$ decrease with age to values lower than λ, cf. $\underline{\psi}$ in Figure 2. The model's convergence property hence implies that survival rates are overestimated eventually even when the initial degree of ambiguity, δ, is low. Overestimation at old age may result from the fact that people have survived the gamble against death several times before. Consequently, one possible heuristic interpretation of age-increasing δ_h might be that "people want to avoid a realistic assessment of their encounter with death".[9]

The concept of *likelihood-insensitivity*, introduced by Peter Wakker and coauthors (cf., Wakker 2004; 2010; Abdellaoui et al. 2011), may provide an alternative heuristic interpretation for the age-increasing δ_h of our model. These authors interpret δ_h not as an ambiguity but rather as a cognitive parameter which reflects the empirical observation that people do not sufficiently distinguish between non-degenerate probabilities. E.g., an extreme example for likelihood insensitivity are "fifty-fifty" probability assessments for any uncertain event and its complement. Under this cognitive interpretation, likelihood insensitivity—and not necessarily ambiguity—would increase with age. Given that old people increasingly suffer from cognitive impairments, this alternative interpretation

[9] This interpretation is consistent with the observation of Kastenbaum (2000) who summarizes the insights of psychological research on the reflection about personal death as follows: "There are divergent theories and somewhat discordant findings, but general agreement that most of us prefer to minimize even our cognitive encounters with death."

has some intuitive appeal. Despite this, we continue to interpret δ_h as age-dependent ambiguity in the remainder of our analysis.

Regardless of the specific interpretation, we can conclude, at this point, that (i) Proposition 1 is derived from sound (albeit strong) decision-theoretic assumptions and (ii) the calibration of our parsimonious belief model will result in a nice fit to the HRS data on subjective survival beliefs (cf. Section 5.1).

3 Quantitative Life-Cycle Model

This section merges our notion of ambiguous survival beliefs with a life-cycle model where one period corresponds to one age year. Households live up to some maximum age, T. We also model a realistic life-cycle income profile including stochastic and age-specific labor productivity. In addition, a PAYG pension system is modeled assuming a fixed date of retirement. We assume no annuity markets and a borrowing constraint. These elements are included only in order to generate realistic endogenous life-cycle consumption profiles. Borrowing constraints, stochastic labor income in combination with impatience gives a hump-shaped consumption profile as we see it in the data. Positive pension income implies that savings for retirement are not too large.

3.1 Demographics

We consider a large number of ex-ante identical agents (=households). Households become economically active at age (or period) 0 and live at most until age T. The number of households of age t is denoted by N_t. Population is stationary and we normalize total population to unity, i.e., $\sum_{t=0}^{T} N_t = 1$. Households work full time during periods $1, \ldots, t_r - 1$ and are retired thereafter. The working population is $\sum_{t=0}^{t_r-1} N_t$ and the retired population is $\sum_{t=t_r}^{T} N_t$.

We refer to age $h \leq t$ as the planning age of the household, i.e., the age when households make their consumption and saving plans for the future. At ages $h = 1, \ldots, T$, households face objective risk to survive to some future period t. We denote corresponding objective survival probabilities for all in-between periods k, $h \leq k < t$, by $\psi_{k,t}$ where $\psi_{k,t} \leq 1$ for all $t \leq T$ and $\psi_{k,t} = 0$ for $t = T + 1$. We think of survival risk as an idiosyncratic risk which washes out at the aggregate level. Total population is therefore constant and dynamics of the population are correspondingly given by $N_{t+1} = \psi_{t,t+1} N_t$, for N_0 given.

3.2 Endowments

There are discrete shocks to labor productivity in every period $t = 0, 1, ..., , t_r - 1$ denoted by $\eta_t \in E$, E finite, which are i.i.d. across households of the same age. The reason for stochastic labor productivity in our model is to impose discipline on calibration. For sake of comparability, our fully rational model features standard elements as used in numerous structural empirical studies (cf., Laibson et al. 1998; Gourinchas and Parker 2002 and references therein). By $\eta^t = (\eta_1, \ldots, \eta_t)$ we denote a history of shocks and $\eta^t \mid \eta^h$ with $h \leq t$ is the history $(\eta_1, \ldots, \eta_h, ..., \eta_t)$. Let E be the powerset of the finite set E and E^{t_r-1} be the σ-algebra generated by $\mathsf{E}, \mathsf{E}, ...$ We assume that there is an objective probability space $\left(\times_{t=0}^{t_r-1} E, \mathsf{E}^{t_r-1}, \pi\right)$ such that $\pi_t(\eta^t \mid \eta^h)$ denotes the probability of η^t conditional on η^h.

In addition, we assume productivity to vary by age where ϕ_t denotes age-specific productivity which will be estimated from the data and results in a hump-shaped life cycle earnings profile.

After retirement at age t_r households receive a lump-sum pension income, b. Retirement income is modeled in order to achieve a realistic calibration. Pension contributions are levied at contribution rate τ.

Collecting elements, income of a household of age t is given by

$$y_t = \begin{cases} \eta_t \phi_t w (1 - \tau) & \text{for } t < t_r \\ b & \text{for } t \geq t_r. \end{cases}$$

There are no annuity markets, an assumption which can be justified by the observed small size of private annuity markets.[10] We assume a fixed zero borrowing constraint and a fixed interest rate r. With cash-on-hand given as $x_t \equiv a_t (1 + r) + y_t$ the budget constraint writes as

$$x_{t+1} = (x_t - c_t)(1 + r) + y_{t+1} \geq 0.$$

Define total income as $y_t^{tot} = y_t + r a_t$, saving as $s_t = y_t^{tot} - c_t$ and gross savings as assets tomorrow, a_{t+1}.

3.3 Government

We assume a pure PAYG public social security system. Denote by χ the net pension benefit level, i.e., the ratio of pensions to net wages. The government budget is assumed

[10] See Friedman and Warshawsky (1990). Observe that underestimation of survival beliefs extenuates the annuity puzzle.

to be balanced each period and is given by

$$\tau w \sum_{t=0}^{t_r-1} \phi_t N_t = b \sum_{t=t_r}^{T} N_t = \chi (1-\tau) w \sum_{t=t_r}^{T} N_t. \tag{1}$$

In addition, accidental bequests—arising because of missing annuity markets—are taxed away at a confiscatory rate of 100%.[11]

3.4 CEU Preferences

Households face two dimensions of uncertainty, respectively risk, about period t consumption. First, due to our assumption of productivity shocks, agents face a risky labor income. Second, agents are uncertain with respect to their life expectancy. While we model income risk in the standard objective EU way, we model uncertainty about life-expectancy in terms of a CEU agent who holds ambiguous survival beliefs as stated in Proposition 1.

Given the productivity shock history η^h, denote by $\mathbf{c} \equiv (c_h, c_{h+1}, c_{h+2}...)$ a shock-contingent consumption plan such that the functions c_t, for $t = h, h+1, ...$, assign to every history of shocks $\eta^t | \eta^h$ some amount of period t consumption. Denote by $u(c_t)$ the agent's utility from consumption at age t. We assume that utility is strictly increasing in consumption and that the agent is strictly risk-averse, i.e., $u'(c_t) > 0$, $u''(c_t) < 0$. Expected utility of an h-old agent from consumption in period $t > h$ contingent on the observed history of productivity shocks η^h is then given as

$$E_t [u(c_t)] \equiv E_t \left[u(c_t), \pi \left(\eta^t | \eta^h \right) \right] = \sum_{\eta^t | \eta^h} u(c_t) \pi \left(\eta^t | \eta^h \right)$$

where we introduce $E_t[\cdot]$ as a shortcut notation for the expectation operator with respect to productivity shock η^t in period t, conditional on period h.

We assume additive time-separability and we add a raw time discount factor $\beta = \frac{1}{1+\rho}$.[12] For any $s \in \{h, h+1, ..., T\}$ and survival until period s, the agent's von Neumann Morgenstern utility (vNM) from a consumption plan $\mathbf{c}$ is then defined as

$$U(\mathbf{c}(s)) = u(c_h) + \sum_{t=h+1}^{s} \beta^{t-h} E_t [u(c_t)].$$

To model survival uncertainty of an agent of age h with respect to ambiguous survival beliefs, we use the sequence of conditional neo-additive probability spaces $(\Omega, \mathcal{F}, \nu(\cdot \mid h))$,

[11] Revenue from this source is used for government consumption which is otherwise neutral.

[12] In line with Halevy (2008) and Andreoni and Sprenger (2012), we assume that time-preferences cannot be reduced to preferences under uncertainty. To keep the formalism as transparent as possible, we simply consider standard exponential time-discounting.

$h = 1, ..., T$, which is mathematically rigorously constructed in Appendix B.3. Denote by $\nu^h \equiv \nu(\cdot \mid h)$ the agent's age-conditional neo-additive capacity and by $\psi_s = \psi(\cdot)$ the objective probability to survive until age s.

In order to formalize utility maximization over life-time consumption with respect to neo-additive probability measures, we henceforth describe an h-old agent as a CEU decision maker who maximizes her Choquet expected utility from life-time consumption with respect to ν^h. By Observation A.1 in Appendix A.1, this agent's CEU from consumption plan $\mathbf{c}$ with respect to ν^h is given as

$$\begin{aligned} E\left[U(\mathbf{c}), \nu^h\right] &= \delta_h \left[\lambda \sup_{s \in \{h, h+1, \ldots\}} U(\mathbf{c}(s)) + (1-\lambda) \inf_{s \in \{h, h+1, \ldots\}} U(\mathbf{c}(s))\right] \\ &\quad + (1-\delta_h) \cdot \sum_{s=h}^{T} \left[U(\mathbf{c}(s)), \psi_s\right]. \end{aligned} \tag{2}$$

The Choquet expected value of a lifetime utility $U(\mathbf{c})$ with respect to a neo-additive capacity ν^h is a convex combination of the expected value of U with respect to some additive probability measure ψ_s and an ambiguity part. In case there is some ambiguity, i.e., $\delta > 0$, parameter λ measures how much weight the decision maker puts on the least upper bound of the range of U. Conversely, $(1-\lambda)$ is the weight she puts on the greatest lower bound. For these bounds we have for any $\mathbf{c}$ that

$$\begin{aligned} \sup_{s \in \{h, h+1, \ldots\}} U(\mathbf{c}(s)) &= u(c_h) + \sum_{t=h+1}^{T} \beta^{t-h} E_t\left[u(c_t)\right], \\ \inf_{s \in \{h, h+1, \ldots\}} U(\mathbf{c}(s)) &= u(c_h), \end{aligned}$$

i.e., the least upper bound consists of the discounted sum of utilities if survival where one in every period while the greatest lower bound in this setting is utility if the agent does not survive to the following period.

Proposition 2. *Consider an agent of age h. The agent's Choquet expected utility from consumption plan $\mathbf{c}$ is given by*

$$E\left[U(\mathbf{c}), \nu^h\right] = u(c_h) + \sum_{t=h+1}^{T} \nu_{h,t}^h \cdot \beta^{t-h} \cdot E_h\left[u(c_t)\right] \tag{3}$$

where the subjective belief to survive from age h to $t \geq h$ is given by

$$\nu_{h,t}^h = \begin{cases} \delta_h \cdot \lambda + (1-\delta_h) \cdot \psi_{h,t} & \text{for } t > h \\ 1 & \text{for } t = h \end{cases}$$

with

$$\delta_h = \frac{\delta}{\delta + (1-\delta) \cdot \frac{1}{1+\sqrt{h}}}.$$

Proof. Fix age h and consider the neo-additive probability space $(\Omega, \mathcal{F}, \nu(\cdot \mid h))$ constructed in Appendix B.3. The objective probability to survive until period t is given as

$$\psi_{h,t} = \prod_{s=h}^{t-1} \psi_{s,s+1}$$

implying

$$\psi_{h,t} = \sum_{s=t+1}^{T} \psi^h(D_s)$$

where D_t denotes the event that the agent dies at the end of period t. Consequently, (2) can be equivalently rewritten as

$$\begin{aligned}
E\left[U(\mathbf{c}), \nu^h\right] = \delta_h & \left(\lambda\left(u(c_h) + \sum_{t=h+1}^{T} \beta^{t-h} E\left[u(c_t), \pi(\eta_t|\eta_h)\right]\right) + (1-\lambda)u(c_h)\right) \\
& + (1-\delta_h)\left(u(c_h) + \sum_{t=h+1}^{T} \psi^h(D_t) \sum_{s=h+1}^{t} \beta^{s-h} E\left[u(c_s), \pi(\eta_s|\eta_h)\right]\right) \\
= {} & u(c_h) + \delta_h \lambda \sum_{t=h+1}^{T} \beta^{t-h} E\left[u(c_t), \pi(\eta_t|\eta_h)\right] \\
& + (1-\delta_h) \sum_{t=h+1}^{T} \psi_{h,t} \cdot \beta^{t-h} E\left[u(c_t), \pi(\eta_t|\eta_h)\right] \\
= {} & u(c_h) + \sum_{t=h+1}^{T} \nu_{h,t}^h \beta^{t-h} E\left[u(c_t), \pi(\eta_t|\eta_h)\right],
\end{aligned}$$

which, together with Proposition 1, proves the proposition.□

3.5 Recursive Problem and Dynamic Inconsistency

In contrast to a sequence of conditional additive probability spaces $(\Omega, \mathcal{F}, \psi(\cdot \mid h))$, $h = 1, ..., T$, our age-dependent sequence of conditional neo-additive probability spaces $(\Omega, \mathcal{F}, \nu(\cdot \mid h))$, $h = 1, ..., T$, (generically) violates dynamic consistency of the agents' life-cycle utility maximization problem whenever $\delta > 0$.[13] To characterize actual behavior

[13] We refer the interested reader to the axiomatic treatment of the relationship between violations of dynamic consistency and violations of Savage's (1954) sure-thing principle (as in CEU theory) to Epstein and Le Breton (1993), Ghirardato (2002), Siniscalchi (2011) and the Appendix in Zimper (2012).

in the presence of dynamic inconsistency, it is convenient to work with the recursive representation of the planning problem under the, arguably, realistic assumption of naive agents. That is, we consider dynamically inconsistent agents who are naive in the sense that they wrongly assume that their optimal consumption plan for any given future-self coincides with this future-self's actual consumption choice.

We further assume that income risk is first-order Markov such that $\pi(\eta^t \mid \eta^{t-1}) = \pi(\eta^t \mid \eta_t)$. It is then straightforward to set up the recursive formulation of lifetime utility (3) for a naive agent. The value function of age $t \geq h$ viewed from planning age h is given by

$$V_t^h\left(x_t, \eta_t\right) = \max_{c_t, x_{t+1}} \left\{ u\left(c_t\right) + \beta \frac{\nu_{h,t+1}^h}{\nu_{h,t}^h} E_t\left[V_{t+1}^h\left(x_{t+1}, \eta_{t+1}\right)\right] \right\}.$$

The naive CEU agent's first order condition is then given by the standard Euler equations.

Proposition 3. *The consumption plan* $\mathbf{c} = (c_h, c_{h+1}, ...)$ *of a naive CEU agent must satisfy, for all* $t \geq h$,

$$\frac{du}{dc_t} \geq \beta\left(1+r\right) \cdot \frac{\nu_{h,t+1}^h}{\nu_{h,t}^h} \cdot E_t\left[\frac{du}{dc_{t+1}}\right] \tag{4}$$

which holds with equality if we have for future asset holdings $a_{t+1} > 0$.

By (4), the expected growth of marginal utility from h to $h+1$ is higher than under rational expectations if the household underestimates the probability of survival to the next period, i.e., if $\nu_{h,h+1}^h < \psi_{h,h+1}$, and vice versa for overestimation.

From the Euler equations (4) we can also directly verify that the CEU life-cycle maximization problem is dynamically inconsistent if and only if the ambiguous survival beliefs do not reduce to additive probabilities. To see this let us compare the optimal consumption choice of an $h+1$ old agent, first, from the perspective of an h old and, second, from her actual perspective when she turns $h+1$. By Proposition 3, the optimal consumption plan for age $h+1$ from the perspective of age h requires (for positive asset holdings) that

$$\frac{du}{dc_{h+1}} = \beta\left(1+r\right) \cdot \frac{\nu_{h,h+2}^h}{\nu_{h,h+1}^h} \cdot E_{h+1}\left[\frac{du}{dc_{h+2}}\right] \tag{5}$$

whereas the optimal consumption choice at age $h+1$ from the perspective of age $h+1$ requires that

$$\frac{du}{dc_{h+1}} = \beta\left(1+r\right) \cdot \frac{\nu_{h+1,h+2}^{h+1}}{\nu_{h+1,h+1}^{h+1}} \cdot E_{h+1}\left[\frac{du}{dc_{h+2}}\right]. \tag{6}$$

Dynamic consistency with respect to the optimal consumption choice at age $h+1$ thus holds if and only if the two first order conditions (5) and (6) coincide. Because of $\nu_{h+1,h+1}^{h+1}=1$, this is the case if and only if

$$\frac{\nu_{h,h+2}^{h}}{\nu_{h,h+1}^{h}}=\nu_{h+1,h+2}^{h+1},$$

which holds for $\delta=0$, implying

$$\frac{\nu_{h,h+2}^{h}}{\nu_{h,h+1}^{h}}=\frac{\psi_{h,h+2}}{\psi_{h,h+1}}=\psi_{h+1,h+2}=\nu_{h+1,h+2}^{h+1},$$

but which is violated for $\delta>0$ since (generically)

$$\frac{\nu_{h,h+2}^{h}}{\nu_{h,h+1}^{h}}=\frac{\delta_h\lambda+(1-\delta_h)\,\psi_{h,h+2}}{\delta_h\lambda+(1-\delta_h)\,\psi_{h,h+1}}\neq\delta_{h+1}\lambda+(1-\delta_{h+1})\,\frac{\psi_{h,h+2}}{\psi_{h,h+1}}=\nu_{h+1,h+2}^{h+1}.$$

As in the static CPT model of Halevy (2008), the CEU life-cycle maximization problem considered in this paper is thus dynamically inconsistent whenever the agents do not reduce to standard RE agents. Whereas dynamic inconsistency in Halevy (2008) results from a fixed non-additive probability weighting function, dynamically inconsistency in our model comes with a sequence of non-additive probability weighting functions.

3.6 Aggregation over Households

Wealth dispersion within each age bin is only driven by productivity shocks. We denote the cross-sectional measure of agents with characteristics (a_t,η_t) by $\Phi_t(a_t,\eta_t)$. Denote by $\mathcal{A}=[0,\infty]$ the set of all possible asset holdings and let $\mathcal{E}$ be the set of all possible income realizations. Define by $\mathcal{P}(\mathcal{E})$ the power set of $\mathcal{E}$ and by $\mathcal{B}(\mathcal{A})$ the Borel σ-algebra of $\mathcal{A}$. Let $\mathcal{Y}$ be the Cartesian product $\mathcal{Y}=\mathcal{A}\times\mathcal{E}$ and $\mathcal{M}=(\mathcal{B}(\mathcal{A}))$. The measures $\Phi_t(\cdot)$ are elements of $\mathcal{M}$. We denote the Markov transition function—telling us how people with characteristics (t,a_t,η_t) move to period $t+1$ with characteristics $t+1,a_{t+1},\eta_{t+1}$—by $Q_t(a_t,\eta_t)$. The cross-sectional measure evolves according to

$$\Phi_{t+1}(\mathcal{A}\times\mathcal{E})=\int Q_t\left((a_t,\eta_t),\mathcal{A}\times\mathcal{E}\right)\cdot\Phi_t\left(da_t\times d\eta_t\right)$$

and for newborns

$$\Phi_1(\mathcal{A}\times\mathcal{E})=N_1\cdot\begin{cases}\Pi(\mathcal{E}) & \text{if } 0\in\mathcal{A}\\ 0 & \text{else.}\end{cases}$$

The Markov transition function $Q_t(\cdot)$ is given by

$$Q_t\left((a_t,\eta_t),\mathcal{A}\times\mathcal{E}\right)=\begin{cases}\sum_{\eta_{t+1}\in\mathcal{E}}\pi\left(\eta_{t+1}|\eta_t\right)\cdot\psi_{t,t+1} & \text{if } a_{t+1}\left(a_t,\eta_t\right)\in\mathcal{A}\\ 0 & \text{else}\end{cases}$$

for all $(a_t, \eta_t) \in Y$ and all $(\mathcal{A} \times \mathcal{E}) \in \mathcal{Y}$.

Aggregation gives average (or aggregate)

$$\begin{array}{ll}
\text{consumption:} & \bar{c}_t = \int c_t(a_t, \eta_t)\Phi_t(da_t \times d\eta_t), \\
\text{assets:} & \bar{a}_t = \int a_t(a_t, \eta_t)\Phi_t(da_t \times d\eta_t), \\
\text{income:} & \bar{y}_t = (1-\tau)w\left(\sum_{t=0}^{t_r-1} \phi_t N_t + \chi \sum_{t=t_r}^{T} N_t\right), \\
\text{total income:} & \bar{y}_t^{tot} = \bar{y}_t + r\bar{a}_t, \\
\text{savings:} & \bar{s}_t = \bar{y}_t^{tot} - \bar{c}_t.
\end{array}$$

4 Calibration

4.1 Household Age

Households enter the model at age 20 (model age 0). The last working year is age 64, hence $t_r = 45$. We set the horizon to some maximum biological human lifespan at age 125, hence $T = 105$. This choice is motivated by Weon and Je (2009) who estimate a maximum human lifespan of around 125 years using Swedish female life-table data between $1950 - 2005$.

4.2 Objective Cohort Data

For objective survival rates we estimate cohort specific survival rates for US cohorts alive in 2007. Objective cross-sectional data is taken from the Social Security Administration (SSA) for $1890 - 1933$ and the Human Mortality Database (HMD) for the years $1934 - 2007$. To obtain complete cohort tables, future survival rates are predicted by the Lee and Carter (1992) procedure. Details are described in Ludwig and Zimper (2013).

Since data on survival rates is unreliable for ages past 100 we estimate survival rates assuming the Gompertz-Makeham law.[14] Accordingly, the mortality rate μ_t at age t is assumed to follow

$$\mu_t = \alpha_1 + \alpha_2 \cdot \exp(\alpha_3 \cdot t) + \epsilon_t, \qquad \epsilon_t \sim \mathcal{N}(0, \sigma^2).$$

We estimate parameters $\{\alpha_i\}_{i=1}^3$ to get an out of sample prediction for ages past 100. The resulting predicted mortality rate function fits actual data very well and is used as objective cohort data in the simulation. According to our estimates, the average mortality rate approaches 1 at ages around age 110 ($t = 90$). For all ages $t = 91, \ldots, 105$, we set the objective survival rate to $\psi_{t,t+1} = \varepsilon = 0.01$.

[14] See, e.g., Preston et al. (2001), p. 192.

4.3 Estimated Subjective Survival Beliefs

We follow Ludwig and Zimper (2013) and estimate parameters $\delta \equiv \delta_{h=0}$ and λ using a non-linear rootfinder to best match the HRS data. Subjective survival rates are obtained by pooling a sample of HRS waves $\{2000, 2002, 2004\}$. Except for heterogeneity in sex and age, we ignore all other heterogeneity across individuals. The estimation yields $\delta = 0.118$ and $\lambda = 0.406$.[15]

4.4 Preferences

We assume a $CRRA$ per-period utility function with $\theta \neq 1$ given by

$$u(c_t) = \Upsilon + \frac{c_t^{1-\theta}}{1-\theta},$$

for all t with preference shifter $\Upsilon \geq 0$. The preference shifter ensures that utility of survival is always higher than utility from death, which is normalized as zero. As a benchmark, we choose $\theta = 3.0$—corresponding to an inter-temporal elasticity of substitution (IES) of one third—and consider as range for sensitivity analysis $\theta \in \{2, 4\}$.

Given $\theta > 1$, per period utility is negative and we therefore calibrate the preference shifter Υ such that condition

$$\sum_{t=h+1}^{T} \beta^{t-h} E_h [u(c_t)] > 0, \text{ for } h = 0, \ldots$$

holds for all t, η^t. We set $\Upsilon = 76.7$ for the naive CEU agent which turns out to be sufficiently high.[16] We further set the discount rate ρ to 5%.

4.5 Prices and Endowments

Wages are normalized to $w = 1$. We consider a symmetric two-state first-order Markov chain for the income process in periods $t = 0, \ldots, t_r$ with state vector $E = [1+\epsilon, 1-\epsilon]$ and symmetric transition matrix $\Pi = [\kappa, 1-\kappa; 1-\kappa, \kappa]$. We take as initial probability vector of the Markov chain $\pi_0 = [0.5, 0.5]'$. Values of persistence and conditional variance of the income shock process are based on the estimates of Storesletten et al. (2004) yielding $\kappa = 0.97$ and $\epsilon = 0.68$.

[15] Estimation results are separately for men and women. We take an equally weighted average of the estimated parameters to get an approximation for λ and δ in the population.

[16] This relates to Hall and Jones (2007) who calibrate—in a different model setup—a preference shifter in the range of $[22.1; 131.9]$. Notice that this is just an arbitrary monotone transformation. Any choice of $\Gamma > 76.7$ ensures that the value of life is always higher than the value of death.

Table 2: Calibrated parameters

Technology and Prices	
$w = 1$	Gross wage
$r = 0.042$	Interest rate
$\tau = 0.124$	Social security contribution rate
$\chi = 0.322$	Net pension benefit level
Income Process	
$\kappa = 0.97$	Persistence of income
$\epsilon = 0.68$	Variance of income
$\{\phi_t\}$	Age specific productivity estimated from PSID
Preferences	
$\theta \in \{2, 3, 4\}$	Coefficient of relative risk aversion
$\rho = 0.05$	Subjective discount rate
$\Upsilon^{CEU} = 76.65$	Preference shifter for naive CEU agent
Subjective Survival Beliefs	
$\delta = 0.118$	Initial degree of ambiguity
$\lambda = 0.406$	Degree of relative optimism
Age Limits and Survival Data	
0	Initial model age (age 20)
$t_r = 45$	retirement (age 65)
$T = 105$	Maximum human lifespan (age 125)
$\{\psi_{k,t}\}$	Objective survival rates from SSA and HMD

Age specific productivity $\{\phi_t\}$ of wages is estimated based on data from the Panel Study of Income Dynamics (PSID) applying the method developed in Hugget et al. (2007). The interest rate is set to $r = 0.042$ based on Siegel (2002). For the social security contribution rate we take the US contribution rate of $\tau = 0.124$. The pension benefit level then follows from the social security budget constraint, cf. equation (1).

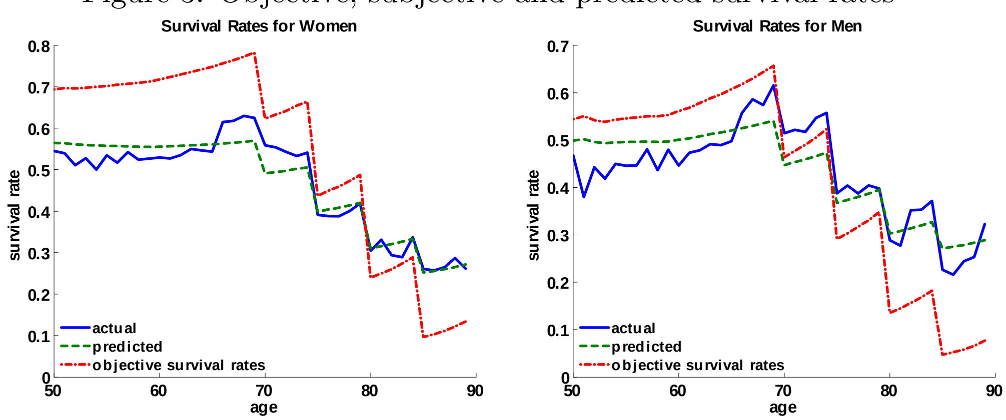
All parameters are summarized in Table 2.

5 Results

5.1 Ambiguous versus Rational Survival Beliefs

Figure 3 compares predicted subjective survival rates resulting from our model of ambiguous survival beliefs with their empirical counterparts and corresponding objective survival rates. Jumps in the figure are due to changes in interview age and respective target age in the survey. Predicted subjective beliefs fit data on subjective survival probabilities well. In particular, the model replicates underestimation of survival rates at younger ages and overestimation at older ages.[17]

Figure 3: Objective, subjective and predicted survival rates

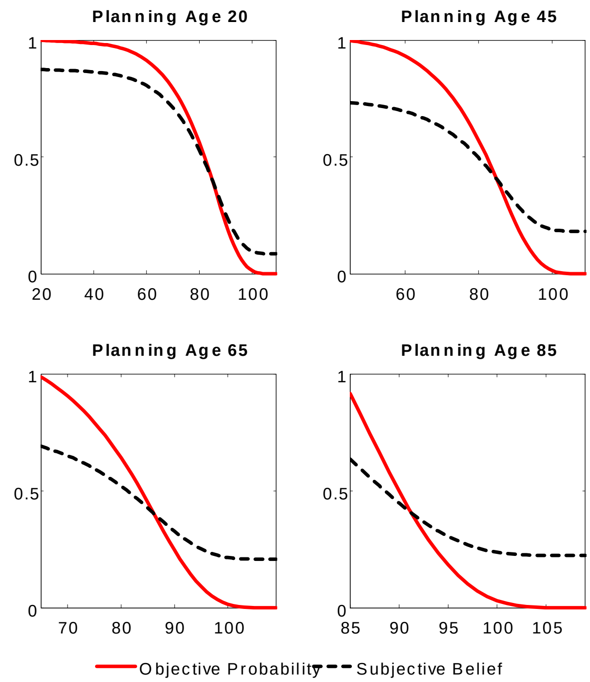


Notes: The figure depicts unconditional survival probabilities to different specific target ages according to the questions in the HRS, cf. Table 1. Interview age is on the abscissa. The figure shows subjective survival beliefs (solid blue line), the corresponding objective survival rates (dashed-dotted red line) and the simulated subjective survival beliefs from the estimated CEU model (dashed green line).

Figure 4 compares the ambiguous survival beliefs at two different planning ages to the objective cohort data. The panels in the figure show unconditional survival rates viewed

[17] Ludwig and Zimper (2013) perform sensitivity analysis with regard to the choice of the initial age, the specific form of the experience function and focal point answers which shows that results do not hinge on these aspects. They also document that biases in beliefs are not due to cohort effects.

Figure 4: Unconditional survival probabilities



Notes: Unconditional objective and subjective probabilities viewed from different planning ages h. Target age is depicted on the abscissa.

from different planning ages where target age t is depicted on the abscissa. Within each of the panels experience, and therefore the ambiguity parameter, does not change. In line with Hammermesh (1985), Peracchi and Perotti (2010), Elder (2013) and several others, subjective survival beliefs generally result in a flatter line than their objective counterparts.

Furthermore, notice that our ambiguous survival beliefs match the stylized fact described by Wu et al. (2013): People at a specific planning (or interview) age underestimate their chances of survival to the nearer future and overestimate the survival probabilities to the more distant future. Also, comparing the different panels in Figure 4 we observe that overestimation of survival probabilities becomes more pronounced as the agent gets older. For example, at age 85 there is underestimation of survival until age 92 while survival to later target ages are overestimated.

We conclude that our calibrated model of ambiguous survival beliefs replicates well the stylized features of the survival belief biases reported in the HRS.

5.2 Life-Cycle Profiles with Ambiguous Beliefs

This subsection compares average plans and realized actions of naive CEU agents. These agents update their plans in each period. As a way to compare any gap between plans made at age h and realizations in $t \geq h$ for CEU agents we denote planned average consumption with superscripts and compute

$$\tilde{c}_t^h = \int c_t^h(a_t, \eta_t)\Phi_t^h(da_t \times d\eta_t) \tag{7}$$

for all t. This gives us hypothetical average consumption profiles in the population if households would stick to their respective period-h plans in all periods $t = h, \ldots, T$. Observe that $\Phi_t^h(\cdot)$ is an artificial distribution generated by respective plans of households. We refer to equation (7) as (average) "planned" consumption (asset, ...) profile in the figures that follow. By dynamic consistency, we have for RE agents that

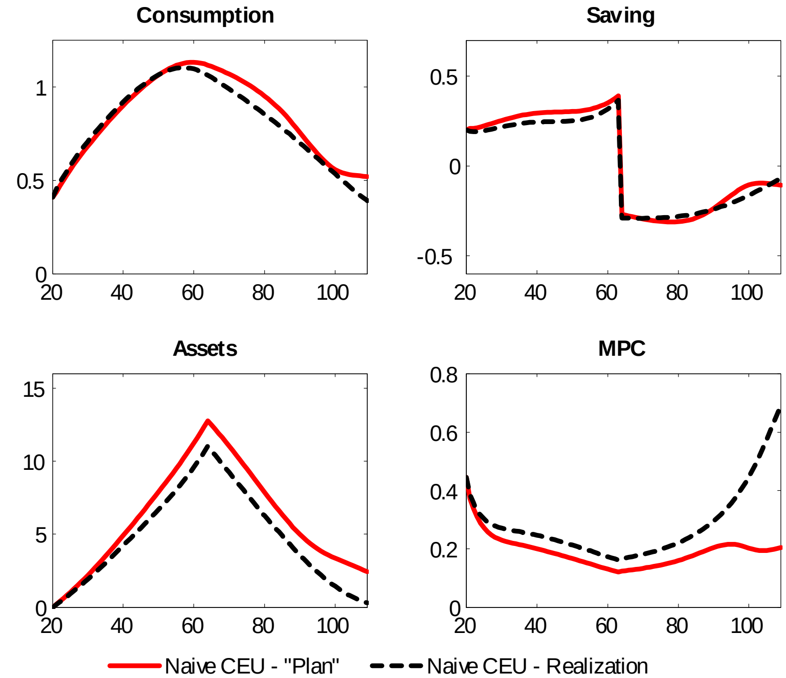
$$c_t^h(a_t, \eta_t) = c_t^1(a_t, \eta_t) \qquad \text{hence} \qquad \tilde{c}_t^h = \tilde{c}_t$$

for all $h = 1, \ldots, T$. These equalities do not hold for naive CEU agents.

Figure 5 compares these objects for naive CEU agents. We compute average consumption, $\tilde{c}_t^h$, net savings, $\tilde{s}_t^h$, assets, $\tilde{a}_t^h$ and total income, $\tilde{y}_t^{tot}$ as well as corresponding average realizations. First note the usual hump-shaped consumption profile, in line with data on non-durable consumption, cf. Fernandez-Villaverde and Krueger (2007).[18] In our calibration we have impatient consumers with $\rho > r$ implying a downward-sloping consumption profile in the retirement period. In the rational expectations model the consumption growth rate reduces more and more as people grow older because of decreasing survival rates. Meanwhile, the CRRA utility function implies consumers to be prudent so that they will save for precautionary motives to self-insure against future income fluctuations. The assumption of a borrowing constraint results in an additional—institution- rather than preference-based—motive for precautionary savings of households to avoid the future potential of binding borrowing constraint. As agents age, motives for precautionary saving become less and less strong and binding constraints less relevant. For these reasons, consumption is initially upward sloping until retirement where the precautionary savings motives are gone. Consequently, the saving rate is positive throughout working life while the agents dissave during retirement. This in turn is reflected in the accumulated assets which rise during working age and peak around retirement entry. Finally, the marginal propensity to consume out of cash-on-hand (MPC) is u-shaped. This results from the interplay of two effects: holding cash-on-hand constant the marginal propensity is low for younger agents and converges to one for older agents, due to

[18] Note that the interpretation of the general shape of the life-cycle profiles also account for the RE agents, displayed in Figure 6.

Figure 5: Average "planned" and realized life-cycle profiles of CEU agents



Notes: Average planned life-cycle profiles of CEU agents at planning age 20 ($h = 1$) compared to average ex-post profiles. MPC denotes the marginal propensity to consume out of cash-on-hand which is approximated by computing averages of $\Delta c/\Delta x$ from the associated policy functions.

their shorter remaining lifetime. For any given age, the marginal propensity decreases with cash-on-hand from values close to one for agents with only little wealth to much lower values for agents with higher cash-on-hand values. As cash-on-hand is low at the beginning of the life-cycle, these effects taken together result in a u-shaped profile.

Comparing the plan and the realization we observe that, initially, CEU agents plan to save more and consume less during working life which would result in higher assets. The planned average saving rate of 20 year old CEU agents is 24.7 percent, whereas the realized saving rate is 21.9 percent. We can thus replicate empirical findings reviewed above that people save less than originally planned, cf. Choi et al. (2006).

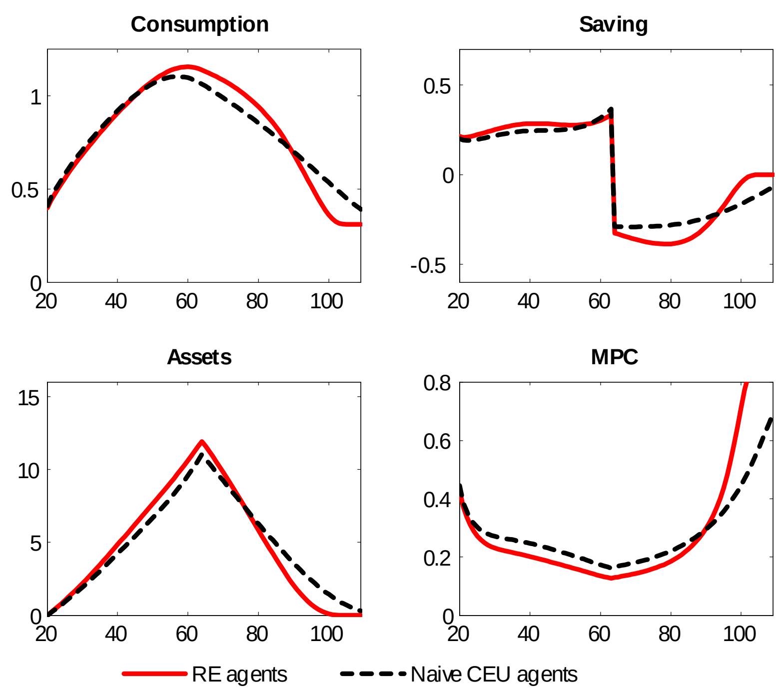
Note also that the planned and realized MPC of CEU agents diverge at older ages. This results from increasing overestimation of survival probabilities which implies large planned old-age asset holdings.

5.3 Life-Cycle Profiles in Rational Baseline Calibration

To highlight the effects of modeling subjective survival beliefs on life-cycle profiles of consumption, saving and asset holdings we compare in Figure 6 naive CEU agents with

RE agents who use objective survival data.[19] On average, CEU agents exhibit undersaving during early working life until age 57 relative to RE agents. Naive CEU agents first consume more than RE agents but start to consume less at age 46 leading to higher saving at age 58 and higher asset holdings later in life. The subjective survival belief model thus gives rise to undersaving at younger ages—due to an underestimation of future survival—and to higher asset holdings at older ages—due to an overestimation of the survival rate at older ages. Correspondingly, the marginal propensity to consume is higher for CEU than for RE agents at younger ages whereas the converse is true at older ages.

Figure 6: Average life-cycle profiles: RE versus naive CEU



Notes: Average life-cycle profile of naive CEU agents compared to RE agents. MPC denotes the marginal propensity to consume out of cash-on-hand which is approximated by by computing averages of $\Delta c/\Delta x$ from the associated policy functions.

Table 3 comprises these results by reporting summary statistics. The average saving rate[20] of CEU agents during their working life is roughly one percentage point lower than the average saving rate of RE agents. More strikingly, average asset holdings of the elderly of ages 85+ are very different between the two types. For CEU agents assets

[19] In this comparison across models we hold the raw discount rate ρ constant. We treat ρ as a deep structural model parameter which is identified by the use of one specific model and calibrated here by reference to our studies. This justifies holding it constant when analyzing counterfactual models.

[20] The saving rate is defined as the ratio of average savings to average income.

Table 3: Summary statistics

		RE	*CEU*
Saving rate[1)]		22.8%	21.9%
Assets at age 85 relative to 65[2)]	Average	35.4%	46.9%
	Median	31.0%	37.3%
Assets at age 95 relative to 65[2)]	Average	8.1%	23.9%
	Median	3.1%	11.6%
Assets of ages 85+ rel. to lifetime average[3)]		52.2%	106.9%

[1)] We define the "average" saving rate as the ratio of averages during working life. We hence compute $\sum s_t / \sum y_t$

[2)] Assets of age 85 (95) relative to assets at retirement entry.

[3)] Percentage difference of average assets during ages 85-110 relative to average assets through whole life.

of the elderly are roughly 107 percent of average assets. On the contrary, for RE agents, average asset holdings of the elderly are only 52 percent of average assets.

Assets of an agent at age 85 (95) relative to her assets at retirement entry are still 46.9% (23.9%) while these values are much lower for RE agents, especially at very old ages. To make our results comparable to the empirical evidence reported by Hurd and Rohwedder (2010), Table 3 also reports numbers on median asset holdings. According to Hurd and Rohwedder (2010), median wealth paths for single households in the HRS indicate that households at age 85 (90) still hold around 49 (21) percent of their assets of age 65. This corresponds to a decumulation speed of $\frac{49}{21} = 2.3$. Our simulation gives a median decumulation speed of RE agents of $\frac{31.0}{3.1} \approx 10.0$ compared to $\frac{37.3}{11.6} \approx 3.2$ for CEU agents which is substantially closer to the empirical Hurd and Rohwedder (2010) benchmark of 2.3. Consequently, ambiguous survival beliefs—resulting in overestimation at old ages—have to be considered as one additional cause for the high old-age asset holdings besides the existing explanations in the form of bequest and precautionary savings motives (De Nardi et al. 2010; Lockwood 2013).

5.4 Trade-off Between Matching Both Empirical Facts

The inter-temporal elasticity of substitution (IES)—the inverse of the coefficient of relative risk aversion θ—influences the willingness to smooth consumption over time. Increasing the IES leads to more consumption at younger ages and to a higher degree of undersaving by the CEU agent. This results in less asset accumulation. In contrast, high old-age asset holdings of CEU agents is less pronounced when the intertemporal substitution elasticity is high. Thus, the choice of the IES determines whether undersaving or high old-age asset holdings is predominant.

Table 4: Summary statistics for different IES[1)]

		High IES		*Low IES*	
		RE	*CEU*	*RE*	*CEU*
Saving rate		19.9%	16.6%	25.0%	24.9%
Assets at 85 rel. to 65	Average	25.1%	25.7%	42.3%	57.7%
	Median	17.3%	9.4%	35.9%	51.7%
Assets at 95 rel. to 65	Average	2.6%	5.0%	13.5%	36.5%
	Median	0.7%	1.0%	9.1%	29.4%
Assets at 85+ rel. to average		31.4%	40.0%	68.7%	145.8%

[1)] High IES is $\theta = 2$, low IES is $\theta = 4$. For a description of how the statistics are constructed see Table 3.

Table 4 shows the saving rate and asset holdings for different values of the IES by setting $\theta \in \{2, 4\}$. In case of a high IES ($\theta = 2$), undersaving by CEU agents increases to 3.3 percentage points. At the same time the difference of average asset holdings of the elderly between CEU and RE are less pronounced. Nevertheless, CEU agents have on average roughly 8.6 percentage points higher relative average assets at old age than RE agents. A lower elasticity ($\theta = 4$) leads to very pronounced high asset holdings of elderly CEU agents which are 77.2 percentage points higher than for average RE agents. The undersaving effect almost vanishes, though. With $\theta = 4$, assets of a 85 (95) year CEU agent relative to the assets at age 65 are at 57.7 (36.5) percent. CEU median asset holdings at age 85 relative to age 65 are 51.7 percent and close to the empirical point estimate of 49 percent, cf. Hurd and Rohwedder (2010). The median CEU decumulation speed is $\frac{51.7}{29.4} \approx 1.75$ which is in fact lower than the empirical benchmark of 2.3.[21] Therefore, our model would exactly replicate this fact with an IES somewhere in the reasonable range of $[0.25, 0.33]$.

6 Concluding Remarks

This paper studies implications of ambiguous survival beliefs for consumption and saving behavior. Point of departure of our analysis is that people make mistakes in assessing their chances to survive into the future: "young" people tend to underestimate whereas

[21] The corresponding median RE decumulation speed is $\frac{35.9}{9.1} \approx 3.94$ which again is far off the empirical benchmark.

"old" people tend to overestimate their survival probabilities. We adopt and parametrize a model of Bayesian learning of ambiguous survival beliefs which replicates these patterns. The resulting conditional neo-additive survival beliefs are merged into a stochastic life-cycle model with CEU (=Choquet expected utility) agents to study life cycle consequences compared to agents with rational expectations (RE).

We show that agents of our model behave dynamically inconsistent. As a result (naive) CEU agents save less at younger ages than they actually planned to save. Due to underestimation of survival at young age, CEU agents also save less than RE agents. Despite this tendency to undersave, CEU agents eventually have higher asset holdings after retirement because of the overestimation of survival probabilities in old age. Overall, our model of mistakes in the assessment of survival prospects adds to explanations for three empirical findings: (i) time inconsistency of agents, (ii) undersaving at younger ages and (iii) high asset holdings at old age. Hence, our model hits at—but does not kill—"three birds with one stone".

Our work gives rise to several avenues of future research. First, observe that the ambiguous survival survival belief functions depicted in Figure 4 closely resemble quasi-hyperbolic time discounting functions, cf., e.g., Laibson (1997). Our ongoing current research compares the qualitative and quantitative features of our ambiguous survival beliefs model with models of hyperbolic time discounting. Of particular interest is the close theoretical (Saito 2011) and empirical (Epper et al. 2011) relationship between the concepts of CPT/CEU and hyperbolic time discounting. Second, we plan to combine our notion of CEU agents with hyperbolic time discounting and/or with bequest and precautionary savings motives in order to cover important aspects of life-cycle decisions. The main challenge for this generalizing approach will be to come up with a parsimonious model in which all calibrated behavioral parameters are identified. Third, we plan to come back in future research to the question of whether "minimizing one's encounter with death" or "age-increasing likelihood insensitivity" (or a combination of both effects) is the better heuristic interpretation for our model's formal feature of an age-increasing δ_h. Finally, we will extend our framework to address normative questions on the optimal design of the tax and transfer system, similar to Laibson et al. (1998), Imrohoroglu et al. (2003) and, more recently, Pavoni and Yazici (2012, 2013) in the hyperbolic time discounting literature.

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A Appendix: Choquet Decision Theory

A.1 Choquet Integration and Neo-additive Capacities

Consider a measurable space $(\Omega, \mathcal{F})$ with $\mathcal{F}$ denoting a σ-algebra on the state space Ω and a non-additive probability measure (=*capacity*) $\kappa : \mathcal{F} \to [0,1]$ satisfying

(i) $\kappa(\emptyset) = 0$, $\kappa(\Omega) = 1$

(ii) $A \subset B \Rightarrow \kappa(A) \leq \kappa(B)$ for all $A, B \in \mathcal{F}$.

The Choquet integral of a bounded $\mathcal{F}$-measurable function $f : \Omega \to \mathbb{R}$ with respect to capacity κ is defined as the following Riemann integral extended to domain Ω (Schmeidler 1986):

$$E[f,\kappa] = \int_{-\infty}^{0} \left(\kappa\left(\{\omega \in \Omega \mid f(\omega) \geq z\}\right) - 1\right) dz + \int_{0}^{+\infty} \kappa\left(\{\omega \in \Omega \mid f(\omega) \geq z\}\right) dz. \quad (8)$$

For example, assume that f takes on m different values such that $A_1, ..., A_m$ is the unique partition of Ω with $f(\omega_1) > ... > f(\omega_m)$ for $\omega_i \in A_i$. Then the Choquet expectation (8) becomes

$$E[f,\kappa] = \sum_{i=1}^{m} f(\omega_i) \cdot \left[\kappa\left(A_1 \cup ... \cup A_i\right) - \kappa\left(A_1 \cup ... \cup A_{i-1}\right)\right].$$

This paper focuses on non-additive probability measures that are defined as *neo-additive capacities* in the sense of Chateauneuf et al. (2007). Recall that the set of *null events*, denoted $\mathcal{N}$, collects all events that the decision maker deems impossible.

Definition 1. *Fix some set of null-events $\mathcal{N} \subset \mathcal{F}$ for the measurable space $(\Omega, \mathcal{F})$. The neo-additive capacity, ν, is defined, for some $\delta, \lambda \in [0,1]$ by*

$$\nu(A) = \delta \cdot \nu_\lambda(A) + (1-\delta) \cdot \mu(A) \quad (9)$$

for all $A \in \mathcal{F}$ such that μ is some additive probability measure satisfying

$$\mu(A) = \begin{cases} 0 & \text{if } A \in \mathcal{N} \\ 1 & \text{if } \Omega \backslash A \in \mathcal{N} \end{cases}$$

and the non-additive probability measure ν_λ is defined as follows

$$\nu_\lambda(A) = \begin{cases} 0 & \text{iff } A \in \mathcal{N} \\ \lambda & \text{else} \\ 1 & \text{iff } \Omega \backslash A \in \mathcal{N}. \end{cases}$$

In this paper, we are exclusively concerned with the empty set as the only null event, i.e., $\mathcal{N} = \{\emptyset\}$. In this case, the neo-additive capacity ν in (9) simplifies to

$$\nu(A) = \delta \cdot \lambda + (1-\delta) \cdot \mu(A)$$

for all $A \neq \emptyset, \Omega$. The following observation extends a result (Lemma 3.1) of Chateauneuf et al. (2007) for finite random variables to the more general case of random variables with a bounded range (see Zimper 2012 for a formal proof).

Observation 1. *Let $f : \Omega \to \mathbb{R}$ be an $\mathcal{F}$-measurable function with bounded range. The Choquet expected value (8) of f with respect to a neo-additive capacity (9) is then given by*

$$E[f, \nu] = \delta(\lambda \sup f + (1-\lambda) \inf f) + (1-\delta) E[f, \mu].$$

A.2 The Generalized Bayesian Update Rule

CEU theory has been developed in order to accommodate paradoxes of the Ellsberg (1961) type which show that real-life decision-makers violate Savage's (1954) sure thing principle. Abandoning the sure thing principle has two important implications for conditional CEU preferences. First, in contrast to Bayesian updating of additive probability measures, there exist several perceivable Bayesian update rules for non-additive probability measures (Gilboa and Schmeidler 1993; Pires 2002; Eichberger et al. 2007; Siniscalchi 2011). Second, if CEU preferences are updated in accordance with an updating rule that universally satisfies the principle of consequentialism, then these CEU preferences violate the principle of dynamic consistency (in a universal sense) whenever they do not reduce to EU preferences (cf. Epstein and Le Breton 1993; Ghirardato 2002; Zimper 2012 and references therein).

In the present paper we assume that the agents form conditional capacities in accordance with the Generalized Bayesian update rule such that, for all non-null $A, B \in \mathcal{F}$,

$$\kappa(A \mid B) = \frac{\kappa(A \cap B)}{\kappa(A \cap B) + 1 - \kappa(A \cup \neg B)}. \tag{10}$$

An application of (10) to a neo-additive capacity ν gives rise to the following observation.

Observation 2. *If the Generalized Bayesian update rule (10) is applied to a neo-additive capacity (9), we obtain, for all non-null $A, B \in \mathcal{F}$,*

$$\nu(A \mid B) = \delta_B \cdot \lambda + (1-\delta_B) \cdot \mu(A \mid B)$$

such that

$$\delta_B = \frac{\delta}{\delta + (1-\delta)\cdot \mu(B)}.$$

B Appendix: Bayesian Learning of Ambiguous Survival Beliefs

This appendix derives Proposition 1 and it constructs the neo-additive probability spaces which we use when we define the life-cycle CEU maximization problem in Section 3.4. To this purpose, let us briefly recall the learning model of ambiguous survival beliefs as introduced in Ludwig and Zimper (2013). We consider an h-old agent, with $0 \leq h \leq k$, who observes the random sample information $\tilde{I}_{n(h)}$ which counts how many individuals out of a sample of size $n(h)$ have survived from age k to t with $k < t$. By assumption, these individuals have the same i.i.d. objective survival probability as the agent.

B.1 The Benchmark Case of Additive Survival Beliefs

At first, consider a standard Bayesian decision maker whose additive estimator for the chance of surviving from k to t conditional on $\tilde{I}_{n(h)}$ is defined as the conditional expected value

$$E\left[\tilde{\theta}, \mu\left(\tilde{\theta} \mid \tilde{I}_{n(h)}\right)\right]$$

where the random variable $\tilde{\theta}$ stands for the agent's survival chance with support on $(0,1)$. By the i.i.d. assumption of individual survivals, $\tilde{I}_{n(h)}$ is, conditional on the true survival probability $\tilde{\theta} = \theta$, binomially distributed with probabilities

$$\mu\left(\tilde{I}_{n(h)} = j \mid \theta\right) = \binom{n(h)}{j} \theta^j (1-\theta)^{n-j} \text{ for } j \in \{0, ..., n(h)\}.$$

We further assume that the agent's prior over $\tilde{\theta}$ is given as a Beta distribution with parameters $\alpha, \beta > 0$, implying $E\left[\tilde{\theta}, \mu\left(\tilde{\theta}\right)\right] = \frac{\alpha}{\alpha+\beta}$. That is, we assume that

$$\mu\left(\tilde{\theta} = \theta\right) = K_{\alpha,\beta}\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

where $K_{\alpha,\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ is a normalizing constant.[22]

[22]The gamma function is defined as $\Gamma(y) = \int_0^\infty x^{y-1}e^{-x}dx$ for $y > 0$.

By Bayes' rule we obtain the following conditional distribution of $\tilde{\theta}$

$$\begin{aligned}
\mu\left(\tilde{\theta}=\theta \mid \tilde{I}_{n(h)}=j\right) &= \frac{\mu\left(\tilde{I}_{n(h)}=j \mid \theta\right)\mu(\theta)}{\int_{(0,1)}\mu\left(\tilde{I}_{n(h)}=j \mid \theta\right)\mu(\theta)\,d\theta} \\
&= K^{\alpha+j-1}_{\alpha+j,\beta+n(h)-k}\theta^{\alpha+j-1}(1-\theta)^{\beta+n(h)-j-1} \text{ for } \theta\in(0,1)
\end{aligned}$$

Note that $\mu\left(\tilde{\theta} \mid \tilde{I}_{n(h)}=j\right)$ is itself a Beta distribution with parameters $\alpha+j, \beta+n(h)-j$. The agent's subjective survival belief conditional on information $\tilde{I}_{n(h)}=j$ is thus given as

$$\begin{aligned}
E\left[\tilde{\theta},\mu\left(\tilde{\theta}\mid j\right)\right] &= \frac{\alpha+j}{\alpha+\beta+n(h)} \\
&= \left(\frac{\alpha+\beta}{\alpha+\beta+n(h)}\right)E\left[\tilde{\theta},\mu\left(\tilde{\theta}\right)\right]+\left(\frac{n(h)}{\alpha+\beta+n(h)}\right)\frac{j}{n(h)}, \\
\text{for } j &\in \{0,...,n(h)\}.
\end{aligned}$$

That is, the posterior estimator $E\left[\tilde{\theta},\mu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right]$ is a weighted average of her prior survival probability $E\left[\tilde{\theta},\mu\left(\tilde{\theta}\right)\right]$, not including any sample information, and the observed sample mean $\frac{j}{n(h)}$.

B.2 Ambiguous Survival Beliefs

Turn now to a Choquet decision maker with neo-additive capacity

$$\nu\left(\tilde{\theta}\right)=\delta\cdot\lambda+(1-\delta)\cdot\mu\left(\tilde{\theta}\right)$$

such that the conditional neo-additive capacity $\nu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)$ results from an application of the Generalized Bayesian update rule. Instead of the additive estimator $E\left[\tilde{\theta},\mu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right]$ we now suppose that the agent's estimator for her survival chance is given as the conditional Choquet expected value

$$E\left[\tilde{\theta},\nu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right]=\delta_{\tilde{I}_{n(h)}}\left(\lambda\sup\tilde{\theta}+(1-\lambda)\inf\tilde{\theta}\right)+\left(1-\delta_{\tilde{I}_{n(h)}}\right)E\left[\tilde{\theta},\mu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right]. \tag{11}$$

For a Beta distribution $\mu\left(\tilde{\theta}\right)$, Ludwig and Zimper (2013) prove the following result:

Observation 3. *The Choquet decision maker's ambiguous survival belief is given as*

$$E\left[\tilde{\theta},\nu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right]=\delta_{\tilde{I}_{n(h)}}\cdot\lambda+\left(1-\delta_{\tilde{I}_{n(h)}}\right)\cdot E\left[\tilde{\theta},\mu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right], \tag{12}$$

with

$$\delta_{\tilde{I}_{n(h)}} = \frac{\delta}{\delta + (1-\delta)\cdot\mu\left(\tilde{I}_{n(h)}\right)}$$

where the unconditional distribution of $\tilde{I}_{n(h)}$ *is given by*

$$\begin{aligned} \mu\left(\tilde{I}_{n(h)} = j\right) &= \binom{n(h)}{j}\frac{(\alpha+j-1)\cdot\ldots\cdot\alpha\cdot(\beta+n(h)-j-1)\cdot\ldots\cdot\beta}{(\alpha+\beta+n(h)-1)\cdot\ldots\cdot(\alpha+\beta)}, \quad (13) \\ \text{for } j &\in \{0,\ldots,n(h)\}. \end{aligned}$$

In a next step, we employ several simplifying assumptions:

Assumption 1. *The additive part* $E\left[\tilde{\theta},\nu\left(\tilde{\theta}\mid\tilde{I}_{n(h)}\right)\right]$ *is, for any information* $\tilde{I}_{n(h)}$*, given as the objective probability, denoted* $\psi_{k,t}$*, to survive from age* k *to* t*.*

Assumption 2. *The agent's additive prior over the parameter space is given as a uniform distribution, i.e., a Beta distribution with parameters* $\alpha=\beta=1$*, implying for (13) that*

$$\begin{aligned} \mu\left(\tilde{I}_{n(h)} = j\right) &= \binom{n(h)}{k}\frac{k!\,(n(h)-k)!}{(n(h)+1)\cdot n(h)!} \\ &= \frac{1}{1+n(h)}. \end{aligned}$$

Assumption 3. *The age-dependent sample-size function is given as*

$$n(h) = \sqrt{h} \text{ for } h \leq T.$$

Assumption 1 is an extreme version of the rational Bayesian learning part of the model developed in Appendix B.1. It specifies a correct additive prior and hence simplifies upon Ludwig and Zimper (2013).[23] By this assumption any difference between subjective survival beliefs and objective survival probabilities are exclusively driven by the ambiguity part of the agent's belief. Assumption 2 allows for an explicit expression of the unconditional probability $\mu\left(\tilde{I}_{n(h)}\right)$ which only depends on age h, i.e., it is identical for every possibly observed sample information $\tilde{I}_{n(h)}$ if h is fixed. By Assumption 3, the agent's experience grows with age but with diminishing returns.

[23] Ludwig and Zimper (2013) are more explicit about the rational Bayesian learning part of the model and assume a proportional bias in prior additive beliefs.

Observation 4. *Under Assumptions 1-3, the estimator (12) simplifies to*

$$\begin{aligned} E\left[\tilde{\theta}, \nu\left(\tilde{\theta} \mid \tilde{I}_{n(h)}\right)\right] &= E\left[\tilde{\theta}, \nu\left(\tilde{\theta} \mid h\right)\right] \\ &= \delta_h \cdot \lambda + (1-\delta_h) \cdot \psi_{k,t} \end{aligned}$$

such that

$$\delta_h = \frac{\delta}{\delta + (1-\delta) \cdot \frac{1}{1+\sqrt{h}}}$$

whenever $h \leq k < t \leq T$, *i.e.,* $\psi_{k,t} \in (0,1)$.

Finally, identifying the h-old agent's subjective belief to survive from k to t with her estimator (11), i.e., defining $\nu_{k,t}^h \equiv E\left[\tilde{\theta}, \nu\left(\tilde{\theta} \mid h\right)\right]$, gives the desired result of Proposition 1.

B.3 Neo-additive Probability Spaces

It remains to provide a mathematically rigorous translation of the notion of ambiguous survival beliefs $\nu_{k,t}^h$ of Proposition 1 into the construction of the conditional neo-additive probability spaces $\left(\Omega, \mathcal{F}^h, \nu(\cdot \mid h)\right)$, $h = 1, ..., T$, that are relevant to the CEU life-cycle maximization problem of Section 3.4.

To this purpose define the finite state space $\Omega = \{0, 1, ..., T\}$ and denote by $\mathcal{F}$ the powerset of Ω. We interpret $D_t = \{t\}$, $t \in \Omega$ as the event in $\mathcal{F}$ that the agent dies at the end of period t. Define age h of the agent as the following event in $\mathcal{F}$: $h = D_h \cup ... \cup D_T$. Further, formally define $Z_{k,t} = D_t \cup ... \cup D_T$ as the event in $\mathcal{F}$ that the agent survives from period k to the beginning of period t.

For each age h, the σ-algebra $\mathcal{F}^h$ is generated by the following partition of Ω: $\{\{0\}, .., \{h-1\}, \{h, ..., T\}\}$. That is, if the agent turns age h she (trivially) observes that she has not died in any previous period but will die at the end of either period h or $h+1$ or ... or T. Observe that our definition of $\mathcal{F}^h$ implies a standard information filtration process because of $\mathcal{F}^1 \subset ... \subset \mathcal{F}^T = \mathcal{F}$.

To conclude the construction of $\left(\Omega, \mathcal{F}^h, \nu(\cdot \mid h)\right)$, $h = 1, ..., T$, define $\nu(Z_{k,t} \mid h) \equiv \nu_{k,t}^h$ such that $\nu_{k,t}^h$ is given by Proposition 1.

C Supplementary Appendix: A Three-Period Model

The intuition of how ambiguous survival beliefs affect consumption and saving behavior will be given in a simple three-period model ($T = 2$) which can be solved analytically. In this simple model we abstract from borrowing constraints, hence $a_{t+1} < 0$, $t < T$ is possible. The no-Ponzi condition $a_{T+1} \geq 0$ is of course assumed. To simplify the analysis we assume the discount factor β to be one and an interest rate r of zero.

As shown in Section 3, lifetime utility for $T = 2$ with ambiguous survival beliefs is expressed as

$$\begin{aligned} U_0^0 &= u(c_0) + \nu_{0,1}^0 u(c_1) + \nu_{0,2}^0 u(c_2) \\ &= u(c_0) + \nu_{0,1}^0 \left(u(c_1) + \frac{\nu_{0,2}^0}{\nu_{0,1}^0} u(c_2) \right), \end{aligned}$$

where $\nu_{k,t}^h$ is the subjective survival belief from Proposition 1. Recall that superscripts denote the respective planning age.

We normalize the utility from death to zero. Lifetime utility of CEU agents reduces to the standard rational expectations case if and only if there is no initial ambiguity, i.e., if $\delta = 0$. As in Section 3 we assume a $CRRA$ per-period utility function with preference shifter $\Upsilon \geq 0$.

We define by $x_t \equiv a_t + y_t$ cash-on-hand as the sum of financial assets a_t and income y_t. In addition, define the present value of future income, $h_t \equiv \sum_{s=t+1}^{T} y_s$, as human wealth. Finally, let total wealth be $w_t \equiv x_t + h_t$. The budget constraint is then given by

$$w_{t+1} = w_t - c_t.$$

In light of the data on subjective beliefs displayed in Figure 1 of the paper we interpret period 0 of the simple model as the period when survival probabilities are underestimated, i.e., up to actual age of about $65 - 70$. Period 1 then reflects the period when there is overestimation in the data. Correspondingly, we make the following assumption:

Assumption 4. *We assume for some $\delta > 0$ that*

$$\psi_{0,1} > \nu_{0,1}^0 = \delta_0 \lambda + (1 - \delta_0)\psi_{0,1} \tag{14}$$

i.e., that $\lambda < \psi_{0,1}$ as well as

$$\psi_{1,2} < \nu_{1,2}^1 = \delta_1 \lambda + (1 - \delta_1)\psi_{1,2} \tag{15}$$

i.e., that $\lambda > \psi_{1,2}$.[24]

Analysis of Consumption and Saving plans

We now turn to the complete inter-temporal household solution to analyze how consumption and saving decisions are altered by biases in subjective survival beliefs.

Rational Expectations

The reference model is the standard solution to the rational expectations model (where $\delta_0 = \delta_1 = 0$). Here, lifetime utility does not depend on the planing period, i.e., $U_1^0 = U_1^1$. Lifetime utility in period 0 is given by $U^0 = u(c_0) + \psi_{0,1}\left(u(c_1) + \psi_{1,2}u(c_2)\right)$.

Observation 5. *Policy functions of the rational expectations solution are linear in total wealth,* $c_t = m_t w_t$*, where*

$$m_t = \begin{cases} \frac{1}{1+\frac{1}{\psi_{t,t+1}^{-\frac{1}{\theta}} m_{t+1}}} & \text{for } t < T \\ 1 & \text{for } t = T. \end{cases}$$

Proof. See, e.g., Deaton (1992) □.

CEU Households

To draw a distinction between RE and CEU households, we use superscript n to denote policy functions (in terms of marginal propensities to consume) of $n$aive CEU households. Given that the household consumes all outstanding wealth in the final period 2 (i.e. $m_2^n = 1$) the solution of the household's problem for all other periods are as follows:

Proposition 4. *The solution for the naive CEU household is as follows:*

- *The solution to the problem in period 1 is:*

$$c_1^{1,n} = m_1^{1,n} w_1 \quad \text{where} \quad m_1^{1,n} = \frac{1}{1 + \frac{1}{\left(\nu_{1,2}^1\right)^{-\frac{1}{\theta}}}}.$$

[24] Notice that, despite equation (15), we may have that the household in period 0 underestimates the probability to survive from period 1 to 2, hence we may have that

$$\psi_{1,2} > \nu_{1,2}^0 = \delta_0 \lambda + (1 - \delta_0)\psi_{1,2}.$$

This is so because $\delta_0 < \delta_1$ and therefore less weight is put on the relative optimism parameter λ.

- *The plan in period 0 for period 1 is:*

$$c_1^{0,n} = m_1^{0,n} w_1 \quad \text{where} \quad m_1^{0,n} = \frac{1}{1 + \frac{1}{\left(\frac{\nu_{0,2}^0}{\nu_{0,1}^0}\right)^{-\frac{1}{\theta}}}}.$$

- *The solution in period 0 is:*

$$c_0^{0,n} = m_0^{0,n} w_0 \quad \text{where} \quad m_0^{0,n} = \frac{1}{1 + \frac{1}{(\nu_{0,1}^0)^{-\frac{1}{\theta}} m_1^{0,n}}}.$$

Proof. Assuming $\beta = R = 1$, the first-order condition in period 1 is:

$$u_c(c_1) = \nu_{1,2}^1 u_c(c_2)$$

which gives

$$c_1^{1,n} = m_1^{1,n} w_1 \quad \text{where} \quad m_1^{1,n} = \frac{1}{1 + \frac{1}{(\nu_{1,2}^1)^{-\frac{1}{\theta}}}}.$$

Period 0: The plan for period 1 gives the first-order condition:

$$u_c(c_1) = \frac{\nu_{0,2}^0}{\nu_{0,1}^0} u_c(c_2)$$

which yields

$$c_1^{1,n} = m_1^{1,n} w_1 \quad \text{where} \quad m_1^{1,n} = \frac{1}{1 + \frac{1}{\left(\frac{\nu_{0,2}^0}{\nu_{0,1}^0}\right)^{-\frac{1}{\theta}}}}$$

The first-order condition in period 0 is:

$$u_c(c_0) = \nu_{0,1}^0 u_c(c_1)$$

yielding

$$c_0 = m_0^{0,n} w_0 = \frac{1}{1 + \frac{1}{(\nu_{0,1}^0)^{-\frac{1}{\theta}} m_1^{0,n}}} w_0.$$

□

We proceed by introducing two definitions which will be used to interpret the above policy functions.

Definition 2 (Moderate Overestimation). *A household* moderately overestimates *planned unconditional survival beliefs if* $\nu_{1,2}^1 < \frac{\nu_{0,2}^0}{\nu_{0,1}^0}$.

Definition 3 (Sufficient Underestimation). *A household at period 0* sufficiently underestimates *objective conditional survival probabilities if* $\frac{\nu^0_{0,1}}{\psi_{0,1}} < \left(\frac{m_1^{1,n}}{m_1}\right)^\theta < 1.$

Comparing the policy functions of Observation 5 and Proposition 4 yields the following Proposition 5 which highlights the consequences of ambiguous survival beliefs for life-cycle savings by studying marginal propensities to consume.

Proposition 5. *The marginal propensities to consume out of total wealth of the CEU agent compared to the RE agent are as follows:*

- *Realization in* $t = 0:$ *Under sufficient underestimation we have that*

$$m_0^{0,n} > m_0.$$

- *Plan for* $t = 1:$ *Under moderate overestimation we have that*

$$m_1^{0,n} < m_1^{1,n}.$$

- *Realization in* $t = 1:$ *We have that*

$$m_1^{1,n} < m_1.$$

At age 0 the realized marginal propensity of the CEU agent, $m_0^{0,n}$, is higher than for an agent with rational expectations, m_0. This outcome only holds under the condition of *sufficient underestimation* of Definition 3. The result implies undersaving, i.e., the naive CEU household saves less out of initial wealth in period 0 than the RE agent.

Turning to the plan of self 0 for the next period 1, observe that the marginal propensity to consume in period 1 planned in period 0, $m_1^{0,n}$, is lower than the realized marginal propensity, $m_1^{1,n}$. Again, this outcome only holds under a certain condition, which is labeled as *moderate overestimation*, cf. Definition 2. That is, only if overestimation is not too large, we can expect model households to save less than originally planned.

Finally, the realized marginal propensity of the CEU agent at age 1, $m_1^{1,n}$, is lower than for the RE agent, m_1. In period 1, the CEU household saves more out of accumulated wealth relative to the RE household. Nevertheless, accumulated wealth is an endogenous object. While it is clear that accumulated wealth of the naive CEU household in period 0 is lower than for an agent with rational expectations, relative wealth positions across the two households in period 1 depend on the relative strength of *sufficient underestimation* in period 0 vis-a-vis *overestimation* in period 1. It is therefore ultimately a quantitative question whether accumulated wealth in period 2 of CEU households exceeds wealth of households with rational expectations.[25]

The analysis of the simple model clarifies that it is a quantitative question whether the calibrated life-cycle model can generate the three empirical regularities on saving behavior: (i) time inconsistent behavior to the effect that people save less than originally planned (under "moderate overestimation"); (ii) undersaving at young age (under "sufficient underestimation"); (iii) high old age asset holdings (if the overestimation eventually outweighs initially low asset accumulation due to the underestimation).

[25] To provide a full characterization we could of course express consumption in all periods as a function of initial wealth. Terms however get messy and interpretation is easier with marginal propensities to consume out of current wealth.