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WARS OF ATTRITION WITH STOCHASTIC COMPETITION

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Wars of Attrition with Stochastic Competition*

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Abstract

We extend the all-pay auctions analysis of Krishna and Morgan (1997) to a stochastic competition setting. In the war of attrition it does not directly follow from the first order condition that the bidding equilibrium strategy is a weighted average of the bidding equilibrium strategies that would be chosen for each number of bidders. This result contrasts with the characterization of the bidding equilibrium strategies in the first-price all-pay auction as well as the winner-pay auctions. Our findings are applicable to future works on contests and charity auctions.

KEYWORDS: Wars of attrition, number of bidders

JEL CLASSIFICATION: D44, D82

1 Introduction

The wide and growing literature on all-pay auctions assumes that the number of bidders is common knowledge. Yet, in many situations where all-pay auctions illustrate economic, social and political issues, participants do not know the number of their opponents. Indeed, in lobbying contests, R&D races or battles to control some markets, agents do not know the exact number of their rivals. In a lobbying contest, some groups of interest give a bribe to the decision maker in order to obtain a market or a political favor. In R&D races, firms compete each other to be the first one to obtain a patent. The money spent in this race is not refundable. More generally, the effect of an unknown number of bidders is an important question in auction theory (see the recent paper of Harstad, Pekec, and Tsetlin (2008)).

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Krishna and Morgan (1997) analyzed these auction designs with affiliated signals where the number of bidders is fixed and common knowledge. In this paper, we extend their analysis to a stochastic competition framework. In the following we call "all-pay auction" the first-price all-pay auction and "war of attrition" the second-price all-pay auction.

McAfee and McMillan (1987) and Matthews (1987) studied first-price auctions with a stochastic number of bidders. They determined if it is better to conceal or to reveal the information about the number of bidders for first and second-price winner-pay auctions in different frameworks.[1] However, they did not characterize the equilibrium strategies. Using a model *à la* Milgrom and Weber (1982) with independent private signals instead of affiliated ones, Harstad, Kagel, and Levin (1990) established that equilibrium bids with stochastic competition are weighted averages of the equilibrium bids in auctions where the number of bidders is common knowledge. Krishna (2002) investigated this result in another way with an independent private value model. In a recent paper Harstad, Pekec, and Tsetlin (2008) found the same result in multi-unit winner-pay auctions with common value.[2]

The equilibrium strategy of the all-pay auction (the proof is omitted), as well as winner-pay auctions (Harstad, Kagel, and Levin, 1990), is a weighted average of equilibrium strategies that would be chosen for each number of bidders. However, it is not obvious for wars of attrition. Indeed, contrary to the – first and second-price – winner-pay auctions, it does not directly follow from the first order condition that the equilibrium strategy should be equal to a weighted average. We provide an answer only for the independent-private-values model.

The paper is organized as follows. The model and preliminaries are given in section 2. In section 3, we analyze the equilibrium strategy in wars of attrition. In section 4, we provide an illustration for the independent private values model. Details of some computations are given in appendix.

2 Model with Stochastic Competition

The model follows and generalizes the preliminaries of Krishna and Morgan (1997) (henceforth K-M) in a stochastic competition setting (as McAfee and McMillan (1987) and Harstad, Kagel, and Levin (1990) used in the study of winner-pay auctions). There is an indivisible object that can be allocated to $N = \{1, 2, ..., n\}$ potential bidders, with $n < \infty$. Every potential bidder is risk neutral. Firstly, we consider a set of bidders $A \subset N$. Denote $|A| = a$ the cardinality of set A.

[1] Matthews (1987) considered bidders with an increasing, a decreasing or a constant absolute risk-aversion and McAfee and McMillan (1987) focused only on the risk-averse bidders and determined the optimal auction.

[2] In their framework, the number of identical prizes is proportional to the number of bidders. They showed that an unknown number of bidders could change the results on information aggregation. Common knowledge of the proportional ratio allows to find the results on information aggregation when the number of bidders is sufficiently high.

Prior to the auction, each bidder i observes a real-valued signal $X_i \in [0, \bar{x}]$. The value of the object to bidder i, which depends on his signal and those of the other bidders, is denoted by

$$V_{a,i} = V_{a,i}(\boldsymbol{X}) = V_a(X_i, \boldsymbol{X}_{-i})$$

where V_a, which is the same function for all bidders, is symmetric in the opponent bidders' signals $\boldsymbol{X}_{-i} = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_a)$. It is assumed that V_a is non-negative, continuous, and non-decreasing in each argument. Moreover, the bidders' valuation for the object is supposed bounded for all a: $\mathbb{E}V_{a,i} < \infty$.

Let f be the joint density of $X_1, X_2, ..., X_a$, a symmetric function in the bidders' signals. Besides, for any a-tuple $y, z \in [0, \bar{x}]^a$ with $\bar{m} = \{\max(y_i, z_i)\}_{i=1}^a$ and $\underline{m} = \{\min(y_i, z_i)\}_{i=1}^a$, f satisfies the affiliation inequality

$$f(\bar{m})f(\underline{m}) \geq f(y)f(z).$$

Affiliation is a strong form of positive correlation as discussed by Milgrom and Weber (1982). It means that if a bidder's signal is high, then other bidders' signals are likely high too. As a consequence, the competition is likely to be strong. Let $F_{Y_a^1}(.|x)$ be the conditional distribution of Y_a^1, where $Y_a^1 = \max\{X_j\}_{j=2}^a$, given $X_1 = x$ and $f_{Y_a^1}(.|x)$ the corresponding density function.

When the number of potential bidders a is common knowledge, we can define

$$v_a(x, y) = \mathbb{E}(V_{a,1}|X_1 = x, Y_a^1 = y), \tag{1}$$

the Bayesian assessment of bidder 1 when his private signal is x and the maximal signal of his opponents is y. As in K-M, we assume that $v_a(x, y)$ is increasing.[3]

We consider the situation in which bidders do not know the number of their rivals when they choose their strategy. For any subset A of N, we denote π_A the probability that A is the set of active bidders. Moreover, the probabilities π_A are independent of the bidders' identities and auction rules. Sets with equal cardinality have equal probabilities. Therefore, the *ex ante* probability to have a participants in the auction is the sum of probabilities with the same cardinal a:

$$s_a := \sum_{|A|=a, A\subset N} \pi_A \text{ such as } \sum_{a=1}^{n} s_a = 1$$

Let p_a^i bidder i's updated probability that there are a bidders conditional upon the event that he is an active bidder. We suppose that these probabilities are common knowledge and symmetric such as $p_a^i = p_a$. Therefore[4]

$$p_a^i := \frac{\sum\limits_{|A|=a, i\in A\subset N} \pi_A}{\sum\limits_{i\in B\subset N} \pi_B} \text{ and } p_a = p_a^i = \frac{a s_a}{\sum\limits_{i=1}^{n} i s_i}$$

[3] As Milgrom and Weber (1982) and K-M remark, since X_1 and Y_a^1 are affiliated, $v_a(x, y)$ is a non-decreasing function of its arguments. But they adopted the same assumption.

[4] For detail, see McAfee and McMillan (1987).

3 Analysis of the War of Attrition

Assume that the number of bidders is common knowledge and each bidder i bids an amount b_i. Thus, the payoff of the bidder i if $\boldsymbol{b}$ is the vector of bids is

$$U_{a,i}(\boldsymbol{b}, \boldsymbol{X}) = \begin{cases} V_{a,i}(\boldsymbol{X}) - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ \dfrac{1}{\#Q(\boldsymbol{b})} V_{a,i}(\boldsymbol{X}) - b_i & \text{if } b_i = \max_{i \neq j} b_j \\ -b_i & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

where $i \neq j$ and $Q(\boldsymbol{b}) := \{\text{argmax}_i\, b_i\}$ is the collection of the highest bids. Strategies at the symmetric equilibrium are noted β_a when the number of bidders a is known. K-M show that the bidding equilibrium strategy when the bidders are informed about the number of bidders a is

$$\beta_a(x) = \int_0^x v_a(t,t)\lambda(t|t,a)dt \tag{2}$$

where $\lambda(y|x,a) = \dfrac{f_{Y_a^1}(y|x)}{1 - F_{Y_a^1}(y|x)}$ and with the following boundary conditions:

$$\beta_a(0) = 0 \text{ and } \lim_{x \to \bar{x}} \beta_a(x) = \infty.$$

Let us assume the same mechanism for a stochastic number of bidders and denoted $\beta^i : [0, \bar{x}] \to \mathbb{R}_+$ a bidder's i pure strategy, mapping signals into bids. As we consider only the symmetric equilibria, we focus on the symmetric pure strategies $\beta \equiv \beta^1 = \beta^2 = ... = \beta^a$. As the number of bidders is stochastic, the definition of the equilibrium strategy concerns bidders' beliefs about the number of active bidders. Strategy β is called a equilibrium strategy if for all bidders i

$$\beta(x) \in \text{argmax}_{b_i}\, \mathbb{E}_a \mathbb{E}[U_{a,i}(b_i, \boldsymbol{\beta(X_{-i})}, \boldsymbol{X})|X_i = x] \;\; \forall x \in [0, \bar{x}] \tag{3}$$

where $\boldsymbol{\beta(X_{-i})} = (\beta(X_1), ...\beta(X_{i-1}), \beta(X_{i+1}), ..., \beta(X_a))$ and $\mathbb{E}_a$ is the expectation operator with respect to the distribution of the bidders' beliefs.

The uncertain number of bidders enters the expected utility through the value of the object for the bidder and the size of the vector of bids $\boldsymbol{b}$.[5]. Assume that all bidders except bidder 1 follow a symmetric – and differentiable – equilibrium strategy. Bidder 1 receives a signal x and bids an amount b. The expected utility of bidder 1 is

$$\begin{aligned} \Pi(b,x) &= \mathbb{E}_a \mathbb{E}[U_{a,1}(b, \boldsymbol{\beta(X_{-1})}, \boldsymbol{X})|X_1 = x] \\ &= \mathbb{E}_a \mathbb{E}\{[V_{a,1} - \beta(Y_a^1)]\mathbb{1}_{\beta(Y_a^1) \leq b} - b\mathbb{1}_{\beta(Y_a^1) > b}|X_1 = x\} \\ &= \mathbb{E}_a \mathbb{E}\{\mathbb{E}\{[V_{a,1} - \beta(Y_a^1)]\mathbb{1}_{\beta(Y_a^1) \leq b} - b|X_1, Y_a^1\}|X_1 = x\} \\ &= \sum_a p_a \int_0^{\beta^{-1}(b)} [v_a(x,y) - \beta(y))] f_{Y_a^1}(y|x)dy - b\left[1 - \sum_a p_a F_{Y_a^1}(\beta^{-1}(b)|x)\right] \end{aligned} \tag{4}$$

[5]It also enters through the collection of the highest bids $Q(\boldsymbol{b})$ Yet, when $\#Q(\boldsymbol{b}) > 1$ the value of the integral is zero: at least one of the support is an atom. Thus, we do not need to consider it.

with $\beta^{-1}(.)$ the inverse function of $\beta(.)$. The maximisation of (4) with respect to b leads to:

$$\sum_a p_a v_a(x, \beta^{-1}(b)) f_{Y_a^1}(\beta^{-1}(b)|x) \frac{1}{\beta'(\beta^{-1}(b))} - \left[1 - \sum_a p_a F_{Y_a^1}(\beta^{-1}(b)|x)\right] = 0 \qquad (5)$$

At the symmetric equilibrium $b = \beta(x)$, thus (5) yields

$$\begin{aligned} \beta'(x) &= \sum_a \frac{p_a v_a(x,x) f_{Y_a^1}(x|x)}{1 - \sum_i p_i F_{Y_i^1}(x|x)} \\ &= \sum_a w_a(x) \beta_a'(x) \end{aligned} \qquad (6)$$

with the weights

$$w_a(x) = \frac{p_a(1 - F_{Y_a^1}(x|x))}{1 - \sum_i p_i F_{Y_i^1}(x|x)} \qquad (7)$$

By (2) and (6) we know that $\beta(.)$ is increasing. It follows that an equilibrium strategy must be given by

$$\beta(x) = \sum_a w_a(x)\beta_a(x) - \sum_a \int_0^x w_a'(t)\beta_a(t)dt \qquad (8)$$

Thus, we have a necessary condition about the shape of β. We prove that it is indeed an equilibrium strategy under an additional assumption, as stated in the next theorem.

Definition 1. *Let $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $\phi(x, y|a) = v_a(x, y)\tilde{\lambda}(y|x, a)$ where $\tilde{\lambda}(y|x, a) = \dfrac{f_{Y_a^1}(y|x)}{1 - \sum_i p_i F_{Y_i^1}(y|x)}$.*

$\phi(., y|a)$ is the product of $v_a(., y)$, an increasing function, and $\tilde{\lambda}(y|x, a)$, a non-increasing function.[6] Besides, ϕ is equivalent to $v_a(x, y)\lambda(y|x, a)$ defined by K-M when the number of agents a is common knowledge.

Assumption 1. *$\phi(x, y|a)$ is increasing in x for all y.*

Theorem 1. *Under assumption 1, a symmetric equilibrium in a war of attrition is represented by*

$$\beta(x) = \sum_a w_a(x)\beta_a(x) - \sum_a \int_0^x w_a'(t)\beta_a(t)dt$$

with $\beta_a(t)$ and $w_a(t)$ given by (2) and (7).

[6]This fact can be proved in a similar way that the hazard rate $\lambda(y|x, a)$ of the distribution $F_{Y_a^1}(y|x)$ is non-increasing in x.

Proof. First, $\beta(.)$ is a continuous and differentiable function. Indeed, by K-M we know that $\beta_a(.)$ is a continuous and differentiable function. We have to verify the optimality of $\beta(z)$ when bidder 1's signal is x. Using equation (5), we find that

$$\begin{aligned}
\frac{\partial \Pi}{\partial \beta(z)}(\beta(z), x) &= \sum_a p_a v_a(x,z) f_{Y_a^1}(z|x) \frac{1}{\beta'(z)} - 1 + \sum_a p_a F_{Y_a^1}(z|x) \\
&= \frac{1}{\beta'(z)} \left[\sum_a p_a v_a(x,z) f_{Y_a^1}(z|x) - \sum_a p_a v_a(z,z) \tilde{\lambda}(z|z,a)(1 - \sum_i p_i F_{Y_i^1}(z|x)) \right] \\
&= \frac{1}{\beta'(z)} (1 - \sum_i p_i F_{Y_i^1}(z|x)) \sum_a p_a [\phi(x,z|a) - \phi(z,z|a)]
\end{aligned}$$

When $x > z$, as $\phi(x|y,a)$ is increasing in x, it follows that $\frac{\partial \Pi}{\partial \beta(z)}(\beta(z), x) > 0$. In a similar manner, when $x < z$, $\frac{\partial \Pi}{\partial \beta(z)}(\beta(z), x) < 0$. Thus, $\frac{\partial \Pi}{\partial \beta(z)}(\beta(x), x) = 0$. As a result, the maximum of $\Pi(\beta(z), x)$ is achieved for $z = x$. ∎

K-M discussed assumption 1 when the number of bidders is common knowledge. This assumption means that $v_a(.,y)$ increases faster than $\tilde{\lambda}(y|x,a)$ decreases. However, as in wars of attrition with a common knowledge number of bidders, this is not a problem. Indeed, this assumption holds if the affiliation between X and Y_a^1 is not so strong. We give an example below to illustrate this discussion with a stochastic number of bidders.[7]

Example 1. *Let* $f(x) = \frac{2^a}{2^a+1}(1 + \prod_{i=1}^a x_i)$ *on* $[0,1]^a$ *with* X_i *bidder* i*'s signals and let* $a = \{2,3\}$*. Therefore,*

$$\begin{aligned}
f_{Y_2^1}(x,y) &= \tfrac{4}{5}(1 + xy) && \text{on } [0,1]^2 \\
f_{Y_3^1}(x,y_1,y_2) &= \tfrac{16}{9}(1 + xy_1y_2)\mathbb{1}_{y_1 \geq y_2} && \text{on } [0,1]^3
\end{aligned}$$

where Y_a^1 *and* Y_a^2 $(Y_a^1 > Y_a^2)$ *denote the highest order statistics. First of all, we can easily verify that the affiliation inequality given holds. We also assume that* $v_a(x,y) = a(x+y)$*. Then computations lead to*

$$\begin{aligned}
f_{Y_2^1}(y|x) &= 2\frac{1+xy}{2+x} && \text{and} \quad F_{Y_2^1}(y|x) = y\frac{2+xy}{2+x} \\
f_{Y_3^1}(y|x) &= 4y\frac{2+xy^2}{4+x} && \text{and} \quad F_{Y_3^1}(y|x) = y^2\frac{4+xy^2}{4+x}
\end{aligned}$$

We can also verify that $F_{Y_a^1}(y|x)$ *is non-increasing in* x*. We obtain*

$$\begin{aligned}
\phi(x,y|2) &= 2(x+y)\frac{2(1+xy)(x+4)}{(x+4)(x+2) - p_2 y(2+xy)(4+x) - p_3 y^2(4+xy^2)(2+x)} \\
\phi(x,y|3) &= 3(x+y)\frac{4y(2+xy^2)(2+x)}{(x+4)(x+2) - p_2 y(2+xy)(4+x) - p_3 y^2(4+xy^2)(2+x)}
\end{aligned}$$

Thus, assumption 1 holds (some details are given in appendix).

[7] This example generalizes an example of K-M with two – fixed – bidders.

Using the results where the number of bidders is common knowledge, the boundary condition $\beta(0) = 0$ follows. Thus, if the expected value is bounded whatever the number of potential bidders, then the bidding strategy will be bounded too. Following the same logic than K-M, we could determine that $\lim_{x \to \bar{x}} \beta(x) = \infty$. Indeed, in this situation,

$$\beta(x) \geq \sum_a p_a \int_0^x v_a(y, y)\tilde{\lambda}(y|y, a)dy + \min_a v_a(z, z) \ln \left(\frac{1 - \sum_a p_a F_{Y_a^1}(z|z)}{1 - \sum_a p_a F_{Y_a^1}(x|z)} \right)$$

If we investigate all-pay auctions with stochastic competition, we would show, under assumption[8] 1, that the equilibrium strategy of all-pay auctions, denoted $\alpha(.)$, is a weighted average of equilibrium strategies, denoted $\alpha_a(.)$, that would be chosen for each number of bidders such as $\alpha(x) = \sum_a p_a \alpha_a(x)$. Thus, the bidders' beliefs about the number of competitors is crucial to determine the bidding strategies. Indeed, the stochastic number of bidders does not modify the bidders' strategies at the equilibrium of all-pay auctions and wars of attrition in the same way.

4 An Example: Independent-Private-Values Model

Harstad, Kagel, and Levin (1990) and Harstad, Pekec, and Tsetlin (2008) show that the form of the equilibrium strategies for winner-pay auctions is such that $\beta(x) = \sum_a w_a(x)\beta_a(x)$. As we seen before, that is still true for the all-pay auction. However, this result is not obvious for the war of attrition. Indeed, contrary to winner-pay auctions and the all-pay auction, in the case of wars attrition, it is not a direct result of the first order condition that the equilibrium strategy should be equal to a weighted average. In this section, we provide an answer only for the IPV model.

Let us consider that each bidder i assigns value X_i to the object, independently distributed on $[0, \bar{x}]$ from the identically distribution F. Therefore, the bidding strategy where the number of bidders a is common knowledge is

$$\beta_a(x) = (a - 1) \int_0^x \frac{yf(y)F^{a-2}(y)}{1 - F^{a-1}(y)} dy$$

and the bidding strategy with stochastic competition is given by

$$\beta(x) = \sum_a p_a (a - 1) \int_0^x \frac{yf(y)F^{a-2}(y)}{1 - \sum_i p_i F^{i-1}(y)} dy$$

.

Lemma 1. *The equilibrium strategy in a war of attrition is decreasing in a for all $a \geq 2$.*

Proof.

$$\frac{\partial \beta_a}{\partial a}(x) = \int_0^x \frac{yf(y)F^{a-2}(y)}{(1 - F^{a-1}(y))^2}[1 - F^{a-1}(y) + (a - 1) \ln F(y)]dy$$

As $1 - F^{a-1}(y) + (a - 1) \ln F(y)$ is negative for all a, y, the result follows. ■

[8] Indeed, this assumption implies that $v_a(., y)f_{Y_a^1}(y|.)$ is increasing for all y. The proof is similar to the proof of the proposition 3 of K-M.

If $\beta(x) \in [\beta_{\underline{a}}(x), \beta_{\bar{a}}(x)]$ for all x with $\beta_{\underline{a}}(x) = \min_a\{\beta_a(x) \forall a \in N | s_a > 0\}$ and $\beta_{\bar{a}}(x) = \max_a\{\beta_a(x) \forall a \in N | s_a > 0\}$ then we can find a vector of weights $(z_a(.))_a$ with $\sum_a z_a(.) = 1, z_a(.) \geq 0$ for all x such that $\beta(x) = \sum_a z_a(x)\beta_a(x)$. Thus, we state:

Proposition 1. *In an IPV model, the equilibrium strategy in wars of attrition with stochastic competition is a weighted average of equilibrium strategies where the number of bidders is common knowledge.*

Proof.

$$\beta(x) - \beta_2(x) = \int_0^x \frac{yf(y)}{[1 - \sum_i p_i F^{i-1}(y)][1 - F(y)]} \left[\sum_a p_a(a-1)F^{a-2}(y) - \sum_a p_a(a-2)F^{a-1}(y) - 1 \right] dy$$

As $\sum_a p_a(a-1)F^{a-2}(y) - \sum_a p_a(a-2)F^{a-1}(y) - 1$ is negative, $\beta(x) \leq \beta_2(x)$.

If $p_1 > 0$, which means that $\beta_{\underline{a}}(x) = \beta_1(x)$, as $\beta_1(x) = 0$ the result follows. However, it is relevant to consider the case $p_1 = 0$:

$$\beta(x) - \beta_n(x) = \int_0^x \frac{yf(y)}{[1 - \sum_{i>1} p_i F^{i-1}(y)][1 - F^{n-1}(y)]} \sum_{a>1} p_a k(y,a) dy$$

where $k(y,a) = (a-1)F^{a-2}(y) + (n-a)F^{n+a-3}(y) - (n-1)F^{n-2}(y)$ is positive for all $a \geq 2$ and y. Hence the result. ∎

5 Conclusion

We have shown that in wars of attrition, it does not directly follow from the first order condition that the equilibrium strategy should be equal to a weighted average. This question remains open for affiliated signals. Even if stochastic competition affects all-pay auctions and wars of attrition in different ways, we could prove – in the same way than K-M – that it does not modify the ranking of the expected revenues.

Our results can be useful for many applications of all-pay designs such as in contest theory and charity auctions. Indeed, recent papers compare all-pay and winner-pay auctions to raise money for charity and suggest to use an all-pay design. In particular, Goeree, Maasland, Onderstal, and Turner (2005) show that the second-price all-pay auction is better to raise money for charity than the first-price all-pay auction and the winner-pay auctions. Charity auctions may be implemented for special events or on the Internet. A large number of charity auctions take place while potential bidders do not know the number of competitors.[9] As we do not introduce externalities in the bidders' payoff, our results could not be applied to charity auctions. However, as they change some insights in the second-price all-pay auctions this work lets us open questions for futur research on charity auctions.

[9] They can know the number of their potential opponents but not the number of their active rivals.

6 Appendix

Boundary Condition of the Equilibrium Strategy.

$$\begin{aligned}
\beta(x) &= \sum_a p_a \int_0^z v_a(y,y)\tilde{\lambda}(y|y,a)dy + \sum_a p_a \int_z^x v_a(y,y)\tilde{\lambda}(y|y,a)dy \\
&\geq \sum_a p_a \int_0^z v_a(y,y)\tilde{\lambda}(y|y,a)dy + \sum_a p_a \int_z^x v_a(z,z)\tilde{\lambda}(y|z,a)dy \\
&\geq \sum_a p_a \int_0^z v_a(y,y)\tilde{\lambda}(y|y,a)dy + \min_a v_a(z,z) \int_z^x \sum_a p_a \tilde{\lambda}(y|z,a)dy \\
&= \sum_a p_a \int_0^z v_a(y,y)\tilde{\lambda}(y|y,a)dy + \min_a v_a(z,z) \ln\left(\frac{1-\sum_a p_a F_{Y_a^1}(z|z)}{1-\sum_a p_a F_{Y_a^1}(x|z)}\right)
\end{aligned}$$

Derivation of the Example.

$$\begin{aligned}
\frac{\partial}{\partial x}\phi^1(x,y|2) =& \frac{4}{(x+2)(1-p_2F_{Y_2^1}(y|x)-p_3F_{Y_3^1}(y|x))}\Bigg[y^2+2xy^2+1-\frac{(x+y)(xy+1)}{x+2} \\
&-(x+y)(xy+1)\frac{-\frac{p_3y^4}{x+4}+\frac{p_3(xy^2+4)y^2}{(x+4)^2}-\frac{p_2y^2}{x+2}+\frac{p_2(xy^2+2y)}{(x+2)^2}}{1-p_2F_{Y_2^1}(y|x)-p_3F_{Y_3^1}(y|x)}\Bigg]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x}\phi^1(x,y|3) =& \frac{12y}{(x+4)(1-p_2F_{Y_2^1}(y|x)-p_3F_{Y_3^1}(y|x))}\Bigg[y^3+2xy^2+2-\frac{(x+y)(xy^2+2)}{x+4} \\
&-(x+y)(xy^2+2)\frac{-\frac{p_3y^4}{x+4}+\frac{p_3y^2(xy^2+4)}{(x+4)^2}-\frac{p_2y^2}{x+2}+\frac{p_2y(xy+2)}{(x+2)^2}}{1-p_2F_{Y_2^1}(y|x)-p_3F_{Y_3^1}(y|x)}\Bigg]
\end{aligned}$$

Computations lead to non-negative derivatives.

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