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# INFLATION AND INNOVATION-DRIVEN GROWTH

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# Inflation and Innovation-driven Growth

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**Abstract**

This paper models the relationship between inflation and steady state growth in a model combining standard Schumpeterian growth with a standard New Keynesian specification of nominal price rigidity. Positive money growth has two clear-cut countervailing effects on the incentive to innovate. Past price rigidity causes the use of an inefficiently large quantity of cheap old intermediate goods, reducing demand for new ones and hence, the incentive to innovate. Future price rigidity erodes the new good's relative price, increasing demand and therefore the current incentive to innovate. In numerical calibrations the negative effect of inflation on growth dominates.

Keywords: Inflation, endogenous growth, price rigidity.
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## 1 Introduction

The effects of money on the real side of the economy remain a field of heated discussion in economics and politics. In particular, price stability is one of the key aims of monetary policy in developed countries where the need for low inflation is often justified citing a conjectured detrimental effect of inflation on economic growth.

A host of papers empirically investigates the relationship between the growth rate of money supply or the inflation rate[1] and the economic growth rate in the long run.[2] Earlier studies, using mostly cross-country data, found no significant correlation between

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[1] Virtually all studies find a very high correlation between money growth and inflation, so that both variables can be used to study the superneutrality question. For example, McCandless and Weber [1995] report a long-run correlation of money growth and consumer-price inflation of 0.925 (0.958) for M0 (M1) in a cross section of 110 countries.

[2] Temple [2000] contains a survey of recent empirical contributions as well as a discussion of the methodological difficulties involved. Summaries of empirical investigations are also given in Ghosh and Phillips [1998], Bruno and Easterly [1998], Ragan [1998] and Briault [1995].

monetary variables and economic growth.[3] More recently, a consensus seems to have emerged that inflation is detrimental to growth. Evidence of this negative relationship has been found in a set of studies making use mainly of panel regression models, such as Barro [1996], Judson and Orphanides [1999], Gylfason and Herbertsson [2001] and Gillman, Harris and Mátyás [2004].[4] Barro [1996], for example, using 10-year averages of data for over 100 countries spanning the period 1960-1990, finds that a 10 percentage point increase in the monetary growth rate decreases economic growth by 0.2-0.3 percentage points, depending on the monetary variable used.[5] Unlike Barro, who cannot reject a linear inflation-growth relationship, most recent studies furthermore find evidence of non-linearity in the data.[6]

The effects of money on output levels and growth have also been investigated extensively in the theoretical literature. The short-run effects of money on output have been fruitfully analysed in the Dynamic Stochastic General Equilibrium framework of the New Neoclassical Synthesis (NNS) business cycle models.[7] At the same time, growth theorists have been interested in understanding the effects of money growth on long-run economic growth. While this issue has been extensively explored in the exogenous growth framework of Ramsey [1928] and Solow [1956],[8] it was the development of endogenous growth theory

[3]The cross-country study of McCandless and Weber [1995] is a good example and contains further references. More recently, Judson and Orphanides [1999] find no significant relation in cross-country data but a negative relation when panel data are used. In a time–series setup, Geweke [1986] finds support for the superneutrality hypothesis using a century of annual U.S. data.

[4]In fact, Temple [2000] reports one way of summarizing empirical findings is that a negative correlation might be present in panel data but not in cross-country data.

[5]The growth rates of M1, M2 and a consumer price index are used as explanatory variables.

[6]Papers investigating the non-linearity of the inflation-growth relationship include Sarel [1996], Ghosh and Phillips [1998], Judson and Orphanides [1999], Gylfason and Herbertsson [2001], Khan and Senhadji [2001], Gillman, Harris and Mátyás [2004] and Burdekin et al [2004]. In particular, the effect of inflation seems to be negative and insignificant or even significantly positive at low levels of inflation, becoming significantly negative once a threshold level has been passed. The threshold is found to be different for industrialized countries, with estimates as low the 1% found by Khan and Senhadji [2001], and developing countries. Some recent estimates for developing countries are 3% (Burdekin et al [2004]) and 11% (Khan and Senhadji [2001]). Khan and Senhadji [2001]) emphasise that the theoretical mechanisms underlying the relationship may be different for industrialized and developing countries, too.

[7]Cf. e.g. to Clarída, Galí and Gertler [1999], Chari, Kehoe and MacGrattan [2000] and Cooley and Hansen [1995]. Important empirical studies investigating the behaviour of money and output at business cycle frequencies include Stock and Watson [2000] and Christiano, Eichenbaum and Evans [2000].

[8]Seminal contributions include Tobin [1965], Sidrauski [1967], Fischer [1979] and Stockman [1981]. This literature is surveyed nicely in Orphanides and Solow [1990]. Of course, money growth could influence economic growth only on the transition path to steady state equilibrium in these models. Concerning the steady state, the models investigate the existence of level effects of inflation on the capital stock and output.

that recently sparked renewed interest in the relationship between inflation and growth. Recent papers investigate different channels through which inflation can affect the accumulation of (physical or human) capital which is the engine of growth. One strand of the literature assumes that the marginal productivity of investment is affected by inflation, thereby influencing investment and growth. The main mechanism investigated by a number of studies, including Gomme [1993], Jones and Manuelli [1995], Fukuda [1996], Itaya and Mino [2003], Gillman, Harris and Mátyás [2004] and Gillman and Kejak [2005], is the inflation tax on labour supply: If money needs to be held to buy goods, consumption gets more expensive relative to leisure as inflation rises.[9] The reduction of labour supply resulting from substitution can affect growth negatively if the marginal product of physical or human capital depends on employment.[10] In other models, the marginal product of capital is affected by the reallocation of productive resources within the economy: When a credit sector is explicitly modelled and credit is a substitute for money in transactions, inflation leads to the reallocation of resources away from productive uses to this financial sector.[11] Another strand of the literature assumes that the marginal productivity of capital is unchanged by inflation, yet the effective return to investment is affected by inflation. In this spirit, Chari, Jones and Manuelli [1996] and Jones and Manuelli [1995] investigate the consequences of a non-indexed tax system with nominal depreciation allowances. By raising the nominal interest rate, inflation reduces the present value of depreciation allowances, raising the effective tax rate and reducing the after tax return on investment. Similarly, they show that cash reserve requirements on bank deposits which banks use to give investment loans reduce the return on investment with increasing inflation. Thus, a

[9] Money demand is introduced via a cash-in-advance constraint on (all or a subset of) consumption goods in most of the cited papers. Alternatively, Wang and Yip [1992] augment a Lucas [1988]-type model assuming real cash balances are an input to output production, and find that money is superneutral. Chang [2002] extends this model by assuming that money is productive in human capital accumulation, too. He finds that when inflation is positive, firms economise on real cash holdings, which reduces the rate of human capital accumulation and hence, growth.

[10] This is most straightforward in the setup of Jones and Manuelli [1995], where the output production function is $y = k^{\alpha}(hn)^{1-\alpha}$, where $n$ is employment and $h$ is human capital which can be accumulated using output. A given stock of capital $k/h$ (which is unaffected by inflation given that the depreciation rates of $h$ and $k$ are identical) is used more intensively when employment increases, raising the marginal product of capital. Fukuda [1996] and Itaya and Mino [2003] find that under certain transaction technologies and parameter constellations, the effect of decreased labour supply can be to *increase* equilibrium employment and growth when production has increasing returns due to labour externalities, such that the slope of the labour demand curve is positive and bigger than that of labour supply.

[11] This effect is detrimental to growth in Marquis and Reffett [1994] but weakens the negative effect of inflation on growth in Gillman, Harris and Mátyás [2004] and Gillman and Kejak [2005] as in the latter two papers, the increasing use of credit reduces the substitution of leisure for consumption. Cf. to Temple [2000] for references on the interaction of inflation, the financial sector and growth.

negative effect of inflation on growth is established in a variety of frameworks.

The present paper contributes to the literature on inflation and endogenous growth. There are to our knowledge no papers investigating the influence of price rigidities and inflation on growth when the latter is fuelled by productivity gains achieved through research and development. As innovations are probably a more important source of long-term growth than is capital accumulation, we think it is important to understand the potential effects of monetary policy on the incentive to engage in research activities and hence, on economic growth. We therefore discuss the superneutrality issue in the 'Quality Ladder' model developed by Aghion and Howitt [1992] and Grossman and Helpman [1991]. Monopolistically competitive firms produce imperfectly substitutable intermediate goods of different qualities which are combined into the aggregate good in the competitive output sector. Efforts of the research and development sector lead to new designs for improved intermediate goods which then replace existing intermediate goods. To be able to discuss the real effects of nominal variables, we introduce money into the model via the standard assumption that households derive utility from holding real balances, as first introduced by Sidrauski [1967]. Following the standard procedure in both the NNS literature and parts of the literature dealing with the growth effects of inflation,[12] we further add price rigidity to the model. New firms entering the intermediate goods market with an improved good set a price which is fixed thereafter. Thus, the timing of price changes is determined endogenously in a natural way. To be able to investigate the comparative static properties of the model with respect to price rigidity, we further allow existing firms to change their prices infrequently, where price readjustment follows the standard timing structure of Calvo [1983] and Kimball [1995].[13]

In this framework, we first show that money has no real effects when prices are perfectly flexible. With price rigidity, we find that money growth influences real activity through various channels. In the unique steady state equilibrium, we identify two clear-cut countervailing effects of inflation on growth. First, the negative past relative price effect: The rigidity-induced dispersion of prices currently charged by intermediate goods producers biases the output sector's demand towards intermediate goods with old prices, leading to inefficient production of output. The resulting decrease in the equilibrium size of the output sector reduces demand for new intermediate goods. Through this channel of aggregate price distortion, money growth lowers the incentive to innovate and therefore, economic growth. Second, there is a positive future relative price effect: The erosion of the firm's relative price will lead to growing demand for the intermediate good while

[12] Cf. e.g. Maussner [2004].

[13] Kimball's model is a variation of Calvo's model. In his version, firms are assumed to set prices so as to maximize the present value of profits, instead of following a rule of thumb as modelled by Calvo.

its own price will be fixed. This entails higher profits, raising the value of a patent and therefore fuelling economic growth. The two effects are highly intuitive and always have opposite signs. Calibrating the model with standard values from the literature, we find that in numerical examples, the former effect dominates such that money growth reduces economic growth. Both effects are of a sizeable dimension individually, while the net effect on growth is in line with the findings of parts of the above-cited empirical literature.

The rest of the paper is organized as follows: Section 2 describes the model. Section 3 presents the steady state equilibrium the comparative static properties of which are discussed in section 4. In section 5 we calibrate the model and in 6, some preliminary results about local stabilityare presented. Section 7 concludes.

## 2 The model

The New Neoclassical synthesis introduces Keynesian elements such as price rigidity into the standard *Real Business Cycle* model. In the same spirit, we introduce Keynesian features into the standard *Schumpeterian growth* model to analyse the relationship between inflation and long-run growth. The introduction of price rigidity has an impact on the optimal behaviour of nearly all sectors. To still facilitate comparison to both NNS models and growth models with endogenous technological change, we use the most standard specification of both building blocks available. The specification of the real side of our model is closest to Barro/Sala-i-Martin [2003] (henceforth: BS). Concerning the modelling of price rigidity, the *frequency* of price readjustment is partly endogenized in a natural way, while the underlying *structure* of pricing signals follows the standard Calvo [1983]-Kimball [1995 specification. We analyse Rational Expectations Equilibria in this model, where we restrict our attention to steady state equilibria with constant output growth.

**Sketch of the Model** In our model, as in the the recent NNS literature, intermediate goods are the only input in the production of the final good,[14] while labour is used in the production of intermediate goods and in research. Therefore, we first present optimal decisions as a function of the size of the output sector and then use labour market equilibrium to determine this variable.[15] More precisely, we first derive the final good sector's optimal demand for goods from the intermediate goods sector and the latter sector's optimal prices and resulting labour demand in a monopolistically competitive environment under price

[14] Cf. e.g. the models of Chari, Kehoe and McGrattan [2000], Kim [2000], Yun [1996] and Erceg, Henderson and Levin [2000]

[15] In this respect our model differs from BS, where all relevant variables depend directly on aggregate labour $L$ as labour is only used in the production of the final good. Therefore, our steps in solving the model differ somewhat from theirs.

rigidity. By further calculating the market value of a new intermediate good producer at the time of his market entry, we get the price for a patent from the research sector. Free entry implies expected profit from research must be zero. Using the information about the patent price, the zero profit condition gives us the optimal research intensity $\mu$ as a function of the size of the output sector. Using optimal labour demands, we then determine the size of the output sector compatible with labour market equilibrium. With this information, production side equilibrium gives us the optimal research intensity as a function of $L$ and other variables firms take as given. Analysis of the public sector that controls the money supply, of the bond market and of the optimal behaviour of households, which hold utility-yielding real balances, yields the remaining conditions necessary to discuss the general equilibrium.

## 2.1 Final good sector

A perfectly competitive final goods sector assembles the economy's final output good $Y(\tau)$ from a large number $N$ of differentiated intermediate goods according to the sector's production function

$$Y(\tau) = \left[ \sum_{j=1}^{N} \left( q^{k_j(\tau)} x_{k_j}(\tau) \right)^{\frac{\alpha-1}{\alpha}} \right]^{\frac{\alpha}{\alpha-1}} \tag{1}$$

where where $x_{k_j}$ is the amount of sector $j$ type $k_j$ intermediate good used, and $q^{k_j}$ is this type's productivity. We assume that the elasticity of substitution between intermediate goods $\alpha$ is larger than unity. The constant-elasticity-of-substitution form of the aggregator follows Dixit and Stiglitz [1977]. The sector's profits are given by

$$\Pi^Y(\tau) = P(\tau)Y(\tau) - \sum_{j=1}^{N} P_{k_j}(\tau) x_{k_j}(\tau) \tag{2}$$

where $P(\tau)$ is the current price for output and $P_{k_j}(\tau)$ is the price charged for one unit of type $k_j$ sector $j$ intermediate good. Cost minimization leads to firms' optimal demand for intermediate goods, which depends negatively on the type's relative price and the elasticity of substitution between productivity-adjusted intermediate goods, $\alpha$, and positively on its productivity $q^{k_j(\alpha-1)}$ and on aggregate demand.

$$x_{k_j}(\tau) = \left( \frac{P_{k_j}(\tau)}{P(\tau)} \right)^{-\alpha} q^{k_j(\alpha-1)} Y(\tau) \tag{3}$$

Perfect competition prevents firms from making positive profits. Optimal demand for intermediate goods and the zero profit condition determine the output price[16]

[16] For a detailed derivation of both optimal demand and the output price cf. Appendix 1.

$$P(\tau) = \left[ \sum_{j=1}^{N} \left( \frac{P_{k_j}(\tau)}{q^{k_j(\tau)}} \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \tag{4}$$

For later use, we define the economy's aggregate technology index, $Q(\tau)$, as the weighted sum of the productivities $q^{k_j(\tau)}$ associated with each sector's intermediate good

$$Q(\tau) = \left[ \sum_{j=1}^{N} q^{(\alpha-1)k_j(\tau)} \right]^{\frac{1}{\alpha-1}} \tag{5}$$

## 2.2 Intermediate goods sector

There is a number $N$ of intermediate goods, with $N$ going to infinity. Each intermediate good is produced by one firm that bought the blueprint from the corresponding research firms. Good $k_j$ is produced using a linear technology with labour as the only input:

$$x_{k_j}(\tau) = L_{k_j}(\tau) \tag{6}$$

### 2.2.1 An intermediate good producer's pricing problem

The instantaneous profit of a firm producing type $k_j$ consists of the difference between its price $P_{k_j}$ and marginal costs, given by the wage rate $w(\tau)$ determined in the perfectly competitive labour market, times the number of units sold.

$$\Pi_{k_j}(\tau) = \left[ P_{k_j}(\tau) - w(\tau) \right] x_{k_j}(\tau)$$

As intermediate goods are imperfect substitutes in the output production function, intermediate goods producers act in an environment of monopolistic competition and can choose an optimal price subject to the output producing sector's demand function.

We assume the existence of nominal rigidities in the intermediate goods markets: Producers can only adjust their prices infrequently. The standard assumption in the literature due to Calvo [1983] and Kimball [1995] is that at each point in time, with an exogenous flow probability $\beta$ firms receive a signal allowing them to readjust their prices, where the signal is generated by a stochastic Poisson process with parameter $\beta$. In our model, a firm entering the market with an innovative intermediate good naturally choose a price for the new product. As the probability of new inventions being made in the research sector is governed by a Poisson process with endogenously determined parameter $\mu$, this does not change the basic structure of pricing signals received by firms as modelled in the literature while determining the parameter endogenously in a natural way. To be able to conduct comparative static experiments with regard to price rigidity, we further allow existing firms to readjust their prices infrequently, making use of the Calvo-Kimball structure.

Firms thus solve an intertemporal problem to maximise profits: Knowing they will not be able to readjust their prices for some time, they choose an optimal price $P_{k_j}(\tau \left| t_{k_j} = \tau \right.)$ to maximise the present value at the time of market entrance $t_{k_j}$ of their future nominal profits obtained while this price is valid, which we denote by $E\left[V\left(P(\tau \left| t_{k_j} = \tau \right.)\right)\right]$. Profits of all future periods $s$ are weighted with the probability that the firm has not been replaced by a successor and has not received a signal allowing to readjust prices as of time $s$. This leads to a discount factor given by the sum of the nominal interest rate, $i$, the flow probability of being replaced by another firm, $\mu_{k_j}(\tau)$ and the flow probability of receiving a reset signal for the price, $\beta$.[17] In steady state equilibrium, the replacement probability will be constant and the same for all firms, so we set $\mu_{k_j}(\tau) = \mu(\tau) = \mu$. The resulting value is

$$E\left[V\left(P(\tau \left| t_{k_j} = \tau \right.)\right)\right] = \int_{\tau}^{\infty} \widetilde{B} e^{-(i+\mu+\beta)(s-\tau)} \left[P_{k_j}(\tau) - w(s)\right] x_{k_j}(s) ds \tag{7}$$

where the constant $\widetilde{B}$ is part of the probability distribution of the pricing signal. Maximising the value given in (7) with respect to the price $P_{k_j}(\tau \left| t_{k_j} = \tau \right.)$ subject to the output producing firm's demand function (3) leads to the following expression for the optimal price at time $t_{k_j}$ when good $k_j$ is invented:

$$P_{k_j}(\tau \left| t_{k_j} = \tau \right.) = P^*(\tau) = \frac{\alpha}{\alpha - 1} \frac{\int_{\tau}^{\infty} e^{-(i+\mu+\beta)(s-\tau)} w(s) P(s)^{\alpha} Y(s) ds}{\int_{\tau}^{\infty} e^{-(i+\mu+\beta)(s-\tau)} P(s)^{\alpha} Y(s) ds} \tag{8}$$

where we rename the chosen price as $P^*(\tau)$ to indicate that the price is optimal for any firm $j$ choosing a price at time $\tau$, regardless of whether the firm first enters the market or has received a reset signal for its price and regardless of their position on the quality ladder, $q^{k_j}$. Intuitively, the optimal price is forward-looking: The numerator takes account of the future development of cost, with $w(s)$ being marginal cost at time $s$. Analogously, the denominator reflects the future development of demand for the good. As in the standard case without rigidities, the mark-up $\frac{\alpha}{\alpha-1}$ reflects the degree of monopoly power and decreases in the elasticity of substitution between intermediate goods, $\alpha$.

Using steady growth of $w$, $Y$ and $P$ at rate $\omega$, $\gamma$ and $\pi$, respectively, the optimal price reduces to[18]

$$P^*(\tau) = \frac{\alpha}{\alpha - 1} \frac{i + \mu + \beta - \alpha\pi - \gamma}{i + \mu + \beta - \alpha\pi - \gamma - \omega} w(\tau) \tag{9}$$

[17] While the parameter $\beta$ is constant by definition, the interest rate $i$ is constant at steady state. The exact form of the discount factor is derived in Appendix 2.

[18] Convergence of the integral requires $i + \mu + \beta - \alpha\pi - \gamma - \omega > 0$. A detailed derivation can be found in appendix 3. We assume here that only the latest quality is available in each sector. At steady state, this

In a world without rigidities, a monopolistically competitive firm would optimally charge a price $P_{k_j}(\tau) = \frac{\alpha}{\alpha-1} w(\tau)$, where the elasticity of substitution between the imperfectly substitutable goods $\alpha$ determines the markup over marginal cost. In the presence of nominal rigidities, the intermediate goods producers thus choose a higher markup if marginal cost increases over time ($\omega > 0$).[19] This is because during intervals of fixed prices, the price-cost ratio erodes at rate $\omega$. The additional markup term $(i+\mu+\beta-\alpha\pi-\gamma)\,/\,(i+\mu+\beta-\alpha\pi-\gamma-\omega)$ increases in the growth rate of marginal cost, $\omega$. As shown in Appendix 7, in the steady state equilibrium the growth rate of the wage $\omega$ equals the growth rate of money supply, $\psi$. The markup therefore increases with money growth. Further, the present value of a given stream of future profits decreases with the interest rate $i$, while the probability of obsolescence $\mu$ and the flow probability of receiving a reset signal for the price reduce the probability of obtaining these profits (while the current price is valid). Thus, the importance of future profits in the firm's optimization is reduced, the higher $i$, $\mu$ and $\beta$.[20] Consequently, the bigger these variables, the closer is the chosen price to the optimal price without rigidity, i.e. the smaller the additional markup $(i+\mu+\beta-\alpha\pi-\gamma)\,/\,(i+\mu+\beta-\alpha\pi-\gamma-\omega)$. On the other hand, future demand increases in $\alpha\pi+\gamma$ so the size of potential future profits relative to current profits increases in these variables. Thus the higher inflation $\pi$ and output growth $\gamma$, the bigger the term $(i+\mu+\beta-\alpha\pi-\gamma)\,/\,(i+\mu+\beta-\alpha\pi-\gamma-\omega)$. By choosing a higher mark-up, the firm reduces the deviation from optimal price in future periods, which are associated with high demand.

**Equilibrium in the market for intermediate goods** Supply in the market for intermediate goods equals demand as production of each good $j$ is determined by the final good sectors's demand function for good $j$ given in equation (3).

### 2.2.2 An intermediate good producer's market value at market entrance

The value of a patent developed in the R&D-sector will be equal to the market value $E\left(V_{k_j}(\tau)\,\middle|\,t_{k_j}=\tau\right)$ of the firm using the patent. We calculate this market value given by the present value at market entrance in $t_{k_j}$ of all future profits of the firm, taking into account stationary growth of $Y$, $P$ and $w$, the probability of obsolescence before time $s$,

amounts to assuming that $\frac{\alpha}{\alpha-1}\frac{i+\mu-\alpha\pi-\gamma}{i+\mu-\alpha\pi-\gamma-\omega} < q^{(\alpha-1)}$. If this is not the case, monopoly pricing by new firms is not an equilibrium: Incumbent firms would deviate by setting the price $q^{-(\alpha-1)}\frac{\alpha}{\alpha-1}\frac{i+\mu-\alpha\pi-\gamma}{i+\mu-\alpha\pi-\gamma-\omega}w(\tau)-\varepsilon$ with $\varepsilon$ arbitrarily small, thereby attracting all demand for sector $j$ intermediate goods while making non-negative profits. Limit pricing would then obtain. So far, we have only analysed limit pricing for the limiting case $\beta = 0$, where results were qualitatively equivalent to the case with monopoly pricing.

[19] In NNS models without real growth, the optimal price at steady state would be $P^*(\tau) = \frac{\alpha}{\alpha-1}\frac{i+\beta+(1-\alpha)\psi}{i+\beta-\alpha\psi}w(\tau)$.

[20] Note that a *higher* value of $\beta$ corresponds to a *lower* level of price rigidity.

$e^{-\mu(s-\tau)}$, and the development of the firm's price:

$$E\left(V_{k_j}(\tau)\left|t_{k_j}=\tau\right.\right)=q^{k_j(\alpha-1)}Y(\tau)P(\tau)^{\alpha}E\left\{\int_{\tau}^{\infty}e^{-\chi(s-\tau)}P_j(s)^{1-\alpha}ds-w(\tau)\int_{\tau}^{\infty}e^{-(\chi-\omega)(s-\tau)}P_j(s)^{-\alpha}ds\right\} \tag{10}$$

where $\chi=i+\mu-\alpha\pi-\gamma$. In any future period $s>\tau$ where it is still active, the firm's price is still $P^*(\tau)$ if no price reset signal has been received between periods $\tau$ and $s$, which has probability $\left(1-\int_{\tau}^{s}\beta e^{-\beta(s-\theta)}d\theta\right)$. If at least one signal has been received, $P_j(s)$ is equal to $P^*(\theta)$ where $\theta$ is the last period where a reset signal was received. $\theta$ can take on any value between $\tau$ and $s$, weighted with the corresponding probability $\beta e^{-\beta(s-\theta)}$. Using this, the firm's value can be rewritten as

$$\begin{aligned}E\left(V_{k_j}(\tau)\left|t_{k_j}=\tau\right.\right)=&A(\tau)\int_{\tau}^{\infty}e^{-\chi(s-\tau)}\left[\int_{\tau}^{s}\beta e^{-\beta(s-\theta)}P^*(\theta)^{1-\alpha}d\theta+(1-\int_{\tau}^{s}\beta e^{-\beta(s-\theta)}d\theta)P^*(\tau)^{1-\alpha}\right]ds\\&-A(\tau)w(\tau)\int_{\tau}^{\infty}e^{-(\chi-\omega)(s-\tau)}\left[\int_{\tau}^{s}\beta e^{-\beta(s-\theta)}P^*(\theta)^{-\alpha}d\theta+(1-\int_{\tau}^{s}\beta e^{-\beta(s-\theta)}d\theta)P^*(\tau)^{-\alpha}\right]ds\end{aligned} \tag{11}$$

where $A(\tau)=q^{k_j(\alpha-1)}Y(\tau)P(\tau)^{\alpha}$. Evaluating the integrals, the market value is[21]

$$E\left(V_{k_j}(\tau)\left|t_{k_j}=\tau\right.\right)=\left(\frac{q^{k_j}}{Q(\tau)}\right)^{\alpha-1}\frac{Y(\tau)}{Q(\tau)}\left[\frac{P^*(\tau)}{P(\tau)Q(\tau)}\right]^{-\alpha}\frac{w(\tau)}{\alpha-1}\left[1+\frac{\beta}{\chi-(1-\alpha)\omega}\right]\frac{1}{\chi-\omega+\beta} \tag{12}$$

The firm's market value can be interpreted as the present value of an infinite stream of profits. Remember that the present value at time $\tau$ of a an infinite stream of steadily growing profits starting at $\tau$ is $\Pi(\tau)/(R-x)$, where $\Pi(\tau)$ is instantaneous profit at $\tau$, $x$ is the profit growth rate and $R$ is the constant interest rate. Note that in our case instantaneous profit is proportional to the sector's position on the quality ladder $\left[q^{k_j}/Q(\tau)\right]^{\alpha-1}$ and the size of the output sector $Y(\tau)/Q(\tau)$ as demand for the good increases in both variables. The corresponding discount rate for the infinite stream of profits consists of several parts given in the last two terms of (12). Remembering that $\chi=i+\mu-\alpha\pi-\gamma$, the obsolescence-adjusted interest rate, $i+\mu$, features in all parts of the discount rate. Furthermore, as described above, the growth rate of profits has to be subtracted from the interest rate. As prices can only be changed infrequently, the growth rate of profits is different in periods where prices can be changed than in periods where prices are fixed. This explains the composite form of the discount factor: The term to the very right is applicable to intervals where prices are fixed: Demand grows at rate $\alpha\pi+\gamma$ as the firm's relative price erodes at rate $\alpha\pi$ while prices are fixed and output growth is

[21] Derivation of the market value is described in more detail in Appendix 2.

$\gamma$.[22] Unit cost grows at rate $\omega$, leading to the discount rate $i+\mu+\beta-(\alpha\pi+\gamma+\omega)$ for profits.[23] The discount rate for profits in intervals with fixed prices increases in $\beta$ as the duration of these intervals decreases in the probability of receiving a reset signal for the price. The term $1+\beta/\{i+\mu-[\alpha\pi+\gamma+(1-\alpha)\omega]\}$ applies to periods where prices can be changed so both revenue and costs grow at the desired rate $\alpha\pi+\gamma+(1-\alpha)\omega$. The higher $\beta$, the higher the number of periods (in addition to the period of market entrance) where this applies.

#### 2.2.3 The Intermediate Goods sector's labour demand

We next derive the sector's labour demand which will later be used to determine the size of the output sector compatible with labour market equilibrium. Due to the linear production function (6), firm $k_j$'s labour demand is equal to the output producing sector's demand for the firm's good. This is found by plugging the optimal price into the demand function, (3). Aggregation leads to the sum of intermediate good producers' labour demands

$$L^X(\tau)=\frac{Y(\tau)}{Q(\tau)}\sum_{j=1}^{N}\left(\frac{P_{k_j}(\tau)}{P(\tau)Q(\tau)}\right)^{-\alpha}\left(\frac{q^{k_j}}{Q(\tau)}\right)^{\alpha-1} \tag{13}$$

Obviously, aggregate demand for intermediate goods depends negatively on the average relative price of intermediate goods relative to technology. This average price effective at time $\tau$ can be expressed as a weighted average of past optimal prices, where the weights $f(s,\tau)$ refer to the probability that a price valid at time $\tau$ has not been changed since time $s$:

$$\sum_{j=1}^{N}P_{k_j}(\tau)^{-\alpha}=\int_{-\infty}^{\tau}f(s,\tau)\left[P^*(s)\right]^{-\alpha}ds$$

More specifically, the weights reflect the probability that a pricing signal was received *or* an innovation made at time $s$, $(\mu+\beta)$ *and* that no such event took place between times $s$ and $\tau$, $e^{(\mu+\beta)(s-\tau)}$. Thus, we have $f(s,\tau)=(\mu+\beta)e^{(\mu+\beta)(s-\tau)}$. Using this and evaluating the integral, we have[24]

$$L^X(\tau)=\frac{Y(\tau)}{Q(\tau)}\left[\frac{P^*(\tau)}{P(\tau)Q(\tau)}\left(\frac{\mu+\beta}{\mu+\beta-\alpha\omega}\right)^{-\frac{1}{\alpha}}\right]^{-\alpha} \tag{14}$$

[22] Cf. to equation (3) to see that demand growth is proportional to the growth of output and the erosion of relative price.

[23] The term $i+\mu+\beta-(\alpha\pi+\gamma+\omega)$ represents the discount rate for the firm's costs as these have the growth rate $\alpha\pi+\gamma+\omega$. Revenue actually grows at rate $\alpha\pi+\gamma$ so $i+\mu+\beta-(\alpha\pi+\gamma)$ would be the appropriate discount rate. Yet note that the optimal price $P^*(\tau)=\frac{i+\mu+\beta-(\alpha\pi+\gamma)}{i+\mu+\beta-(\alpha\pi+\gamma+\omega)}w(\tau)$ enters the present value of revenue multiplicatively. As this optimal price takes account of the difference between revenue growth and the growth rate of costs, the term $i+\mu+\beta-(\alpha\pi+\gamma)$ cancels out and the discount rate $i+\mu+\beta-(\alpha\pi+\gamma+\omega)$ applies to both revenues and costs.

[24] A more detailed derivation can be found in Appendix 5.

If prices could be reset each period, all firms would charge today's optimal price relative to technology $P^{*}(\tau)/\left(P(\tau)Q\left(\tau\right)\right)$. Thus, the term $\left(\frac{\mu+\beta}{\mu+\beta-\alpha\omega}\right)^{-\frac{1}{\alpha}}$ represents the rigidity-caused deviation of the average relative price from today's optimal relative price. The rigidity-caused expression decreases in $\beta$ and $\mu$, reflecting the fact that the smaller $\mu$ and $\beta$, the higher the probability that prices were set long ago and the smaller the average price. Thus a higher degree of price rigidity (lower $\beta$) leads to higher demand for intermediate goods because they are cheaper on average. Further, as the average price decreases in the wage growth rate $\omega$, demand increases with $\omega$: The higher the growth rate of wages, the faster the change in the optimal price chosen by firms and thus the cheaper are goods with old prices compared to goods with new prices, i.e. the more effective is a given level of price rigidity.[25] As will be shown in section 3.1.1, the wage growth rate $\omega$ equals the rate of money growth $\psi$. Thus we have here one of the mechanisms through which money influences the real side of the economy: Total demand for intermediate goods increases with $\psi$ because given rigidity, money growth reduces the average relative price. The mechanism will be discussed in more detail in the section concerned with comparative statics, 4.

## 2.3 Patents and the R&D sector

In the research and development sector, there are $N$ small research firms, each trying to improve the quality of one existing intermediate good. The flow probability of an invention being made is governed by a Poisson process with parameter $\mu_{k_j}\left(\tau\right)$ for the firm trying to improve intermediate good $k_j$. For a given quality rung $k_j$ (i.e., current position of sector $j$), the probability of success depends linearly on the amount of research labour currently used, $L_j^R(\tau)$:

$$\mu_{k_j}(\tau) = \phi(k_j(\tau))L_j^R(\tau) \tag{15}$$

For any given level of research, the probability of success decreases in the number of innovations that have already been made in that particular sector. This idea is captured by assuming $\phi'(k_j(\tau)) < 0$. In case of success for research firm $j$, the design of the new, improved good will be sold to a new firm that will enter the market for intermediate goods, replacing the incumbent in sector $j$. Given that positive profits are made in the intermediate goods sector, there will be competition for the patent of the new product. Potential firms will bid up the price such that the research firm is able to extract the intermediate good producer's entire profit. A potential producer's maximum willingness to pay is the present value of all future profits at market entrance $E\left[V_{k_j+1}\left(\tau\right)\middle|t_{k_j}=\tau\right]$, as given in equation (12). Given that research firms charge exactly this price, all new

[25] The wage growth rate enters the rigidity term in the form with factor $-\alpha$ because the optimal price *to the power of* $-\alpha$, which is the relevant variable for demand, grows at this rate.

patents will be bought and the market for patents clears.

Thus, sector $j$ research firm's expected profit at time $\tau$ is

$$\Pi_j^R(\tau) = \mu_{k_j}(\tau) E\left[V_{k_j+1}(\tau) \left| t_{k_j} = \tau\right.\right] - w(\tau) L_j^R(\tau)$$

There is free entry into the research sector, so firms' expected profit is zero at every instant which using (15) implies that either no research is undertaken ($L_j^R(\tau) = 0$) or

$$\phi(k_j(\tau)) E\left[V_{k_j+1}(\tau) \left| t_{k_j} = \tau\right.\right] - w(\tau) = 0 \tag{16}$$

holds. Thus, zero profit implies that expected profit from research per researcher equal the wage.

For any given research effort, the probability of making an innovation decreases in the sector's position on the quality ladder. We choose a specification for $\phi(k_j(\tau))$ that implies the existence of spillovers in research: The lower the sector's quality level in comparison to aggregate quality, i.e. the further away the sector is from the research frontier, the easier is making an innovation:

$$\phi(k_j(\tau)) = (\alpha - 1) \frac{1}{\lambda} \left(\frac{q^{k_j+1}}{Q(\tau)}\right)^{1-\alpha} \tag{17}$$

where $1/\lambda$ is the productivity of labour in research. As will become obvious, the specification is chosen such as to exactly offset the positive effect of a sector's position on the quality ladder on expected profit, which is due to positive dependence of the expected market value of the firm using the new patent on its quality. As is a strong but standard assumption in the literature to make sure that the optimal research intensity $\mu$ can be constant and independent of a sector's position.

### 2.3.1 The Research sector's labour demand

Research firm $k_j$'s labour demand is found by rearranging (15) and plugging in $\phi(k_j(\tau))$ as defined in equation (17). Aggregating over all research firms, total demand for research labour thus is

$L^R(\tau) = \sum_{j=1}^N L_{k_j}^R(\tau) = \sum_{j=1}^N \frac{\mu}{\frac{1}{\lambda}\frac{\alpha-1}{q^{\alpha-1}}\left(\frac{q^{k_j(\tau)}}{Q(\tau)}\right)^{1-\alpha}}$ or

$$L^R = \mu\lambda \frac{q^{\alpha-1}}{\alpha - 1} \tag{18}$$

### 2.3.2 Behaviour of the aggregate quality index $Q(\tau)$ and the growth rate

The innovations made in the R&D sector determine the evolution of the quality index defined in equation (5). In case of an innovation occurring in sector $j$, the sector's quality increases from $q^{k_j(\tau)}$ to $q^{k_j(\tau)+1}$. At the same time, the sector's contribution to the aggregate quality index rises from $q^{(\alpha-1)k_j(\tau)}$ to $q^{(\alpha-1)[k_j(\tau)+1]}$. Assuming again that the flow probability of an innovation occurring, $\mu$, is constant and equal across sectors, the expected proportional change at time $\tau$ of sector $j$'s contribution to the index is

$$E(\Delta \text{qual}_\tau^j) = \mu \left( \frac{q^{(\alpha-1)(k_j(\tau)+1)} - q^{(\alpha-1)k_j(\tau)}}{q^{(\alpha-1)k_j(\tau)}} \right) = \mu \left( q^{\alpha-1} - 1 \right) \tag{19}$$

The expected growth rate of the quality index (5) can then be calculated from the sum of these contributions. The Law of large numbers further implies that for a large number $N$ of sectors, the actual growth rate of the quality index, $\gamma_Q$, converges to the expected growth rate:[26]

$$\gamma_Q = E\left[\widehat{Q(\tau)}\right] = \mu \frac{q^{\alpha-1} - 1}{\alpha - 1} \tag{20}$$

It will be shown in section (2.4) that at steady state, the growth rate of output $\gamma$ equals the growth rate of the aggregate quality index, so we have

$$\gamma = \frac{q^{\alpha-1} - 1}{\alpha - 1} \mu \tag{21}$$

## 2.4 Labour market equilibrium

The variables determining the model's production side equilibrium, most notably expected profit in the research sector, depend on the size of the output sector. Using equilibrium in the labour market we now pin down this variable as a function of endogenous variables and employment $L$, as in Sala-i-Martin [2003] and then proceed to the production side equilibrium.

Equilibrium in the labour market requires that the sum of the labour demands of the intermediate goods sector, $L^X(\tau)$, and of the research sector, $L^R$, equal labour supply $L$:

$$L = L^X + L^R \tag{22}$$

Plugging in the labour demands from the intermediate goods sector, (14), and the research sector, (18), determines the size of the final good sector[27]

$$\frac{Y(\tau)}{Q(\tau)} = \frac{L - \mu\lambda \frac{q^{\alpha-1}}{\alpha-1}}{\left[ \frac{P^*(\tau)}{P(\tau)Q(\tau)} \left( \frac{\mu+\beta}{\mu+\beta-\alpha\omega} \right)^{-\frac{1}{\alpha}} \right]^{-\alpha}} \tag{23}$$

As discussed in section 2.2.3, demand for intermediate goods increases in the rigidity-caused term $\frac{\mu+\beta}{\mu+\beta-\alpha\omega}$ as the average price charged by intermediate goods firms, $\frac{P^*(\tau)}{P(\tau)Q(\tau)} \left( \frac{\mu+\beta}{\mu+\beta-\alpha\omega} \right)^{-\frac{1}{\alpha}}$, decreases in this term. Thus, the size of the output sector $\frac{Y(\tau)}{Q(\tau)}$ compatible with labour market equilibrium decreases in the level of rigidity $\beta$ for given $\mu$ and a given size of the labour force $L$.

[26] A detailed derivation can be found in Appendix 6.

[27] Note that from Appendix 7, $\frac{P^*(\tau)}{P(\tau)Q(\tau)}$ is constant. Hence, we have that for constant $\mu$, $\frac{Y(\tau)}{Q(\tau)}$ is constant and $\gamma = \gamma_Q$ at steady state, as asserted above.

### 2.5 Production side equilibrium

Plugging $\phi(k_j(\tau))$ from equation (17) and a new firm's expected market value $E\left(V_{k_j+1}(\tau)\,|t_{k_j}=\tau\right)$ (12) into the zero profit condition (16), we have

$$\frac{1}{\lambda}\frac{Y(\tau)}{Q(\tau)}\left[\frac{P^*(\tau)}{P(\tau)\,Q(\tau)}\right]^{-\alpha}\frac{w(\tau)}{\alpha-1}\frac{\chi+\beta-(1-\alpha)\,\omega}{\chi-(1-\alpha)\,\omega}\frac{1}{\chi+\beta-\omega}=w(\tau)$$

with $\chi=i+\mu-\alpha\pi-\gamma$, which determines the research intensity $\mu$ which makes current research firms indifferent with regard to the amount of research labour used.[28] As can be seen, the resulting research intensity is the same for all firms making an innovation at time $\tau$, regardless of the sector's current position on the quality ladder, consistent with our assumption in section 2.2. The research intensity $\mu$ depends on the size of the final good sector, $\frac{Y(\tau)}{Q(\tau)}$, which we have determined in the equation describing labour market equilibrium, (23). Replacing $\frac{Y(\tau)}{Q(\tau)}$ with this value and rearranging gives us the zero profit condition in the production side equilibrium

$$\frac{\frac{L}{\lambda}-\mu\frac{q^{\alpha-1}}{\alpha-1}}{\frac{\mu+\beta}{\mu+\beta-\alpha\omega}}\frac{i+\mu+\beta-[\alpha\pi+\gamma+(1-\alpha)\,\omega]}{i+\mu-[\alpha\pi+\gamma+(1-\alpha)\,\omega]}\left[i+\mu+\beta-(\alpha\pi+\gamma+\omega)\right]^{-1}=1 \tag{24}$$

### 2.6 Public Sector

The public sector is modelled as in earlier monetary growth models choosing the most parsimonious specification: The state expands the money supply at a constant rate $\psi$ and distributes seigniorage to the households in form of a lump-sum transfer.[29] In particular, the independent central bank perfectly controls the money supply, $M^s(\tau)$, by setting the constant exogenous rate $\psi$:

$$\frac{\dot{M^s}(\tau)}{M(\tau)}=\psi \tag{25}$$

This simple money supply rule is sufficient for our purposes because we are not concerned with either the replication of actual data of central bank behaviour or the design of optimal monetary policy.

All revenue from money creation is allocated to households in form of a lump-sum cash transfer, $T(\tau)$

$$\dot{M^s}(\tau)=T(\tau) \tag{26}$$

The state does not levy taxes and there is no government spending apart from the transfer of seigniorage to households.

[28] Note that the firm's value is $E\left(V_{k_j+1}(\tau)\,|t_{k_j}=\tau\right)$ because it will produce the *next* quality, $k_j+1$ for the sector, which is about to be developed.

[29] Cf. to e.g. Gillman and Kejak [2005], Chang [2002], Marquis and Reffett [1995], Orphanides and Solow [1990].

## 2.7 Bond market

There is a positive number of investment funds in the economy which finance research activities. Each fund is of sufficient size to diversify the risk associated with its investments, such that they only care about the expected profit of each investment.[30] Funds issue a total of $B(\tau)$ bonds which can be bought by households at price $P^B(\tau)$. Given the existence of a risk-free nominal interest rate $i(\tau)$, no-arbitrage implies that the profit a bond yields from interest payments $i^{\text{Bonds}}(\tau)$ and increases in the bond's value, $\dot{P^B}(\tau)$ must equal $i(\tau)P^B(\tau)$:

$$i^{\text{Bonds}}(\tau) = i(\tau)P^B(\tau) - \dot{P^B}(\tau) = \left[i(\tau) - \frac{\dot{P^B}(\tau)}{P^B(\tau)}\right] P^B(\tau) \tag{27}$$

At each moment, a number $\dot{B^s}(\tau)$ of new bonds is issued to finance research and development activities, with

$$P^B(\tau)\dot{B^s}(\tau) = w(\tau)\, L^R(\tau)$$

In return, all returns from selling patents resulting from successful research are allocated to the investment funds. Interest payments sum up to equal these revenues such that investment funds make zero profits.

Equilibrium in the bonds market requires the number of outstanding bonds $B^s(\tau)$ to be equal to the number of bonds $B^d(\tau)$ households wish to hold at price $P^B(\tau)$:

$$B^s(\tau) = B^d(\tau)$$

## 2.8 Households

There is a continuum of households with mass one distributed uniformly on the interval $[0, 1]$. This allows us to interpret aggregate quantities as per capita quantities associated with the decisions of a representative household. The infinitely lived representative household is assumed to maximise the present value of utility over his lifetime, where future flows of instantaneous utility are discounted with the factor $\rho > 0$. The household needs money for transaction purposes. Instead of modelling the transactions services of money explicitly or introducing a cash-in-advance-constraint we adopt the widespread shortcut-assumption that households derive utility from holding real balances $m = \frac{M}{P}$.[31] Letting

[30] At the same time, funds are not big enough to internalize existing distortions.

[31] Feenstra [1986] shows that our case of non-separable utility for consumption and real balances is equivalent to the explicit modelling of cash holdings' transaction cost reducing function, which is the standard justification for choosing this shortcut. This representation is used in many monetary Dynamic General Equilibrium models, such as Chari, Kehoe and McGrattan [2000], Kim [2000], Erceg, Henderson and Levin [2000].

$c$ denote per capita consumption of the final good and assuming the rate of population growth to be zero, a standard specification for households' utility is

$$U = \int_{s=0}^{\infty} e^{-\rho s} \frac{(c(s)^{1-\theta} m(s)^{\theta})^{1-\eta} - 1}{1-\eta} ds \tag{28}$$

where we assume $\eta > 0$, $\theta \in [0, 1)$. The representative household maximises (28) subject to his budget constraint. He inelastically supplies $L$ units of labour at the wage rate $w(\tau)$,[32] receives a transfer $T(\tau)$ from the government and has an interest income of $i^{\text{Bonds}}(\tau)B(\tau)$ from his bond holdings, where $i^{\text{Bonds}}(\tau)$ is given in (27). Income is used for consumption $c(\tau)$ at price $P(\tau)$, the acquisition of new bonds $\dot{B}(\tau)$ at price $P^B(\tau)$[33] and additional money holdings $\dot{M}^d(\tau)$:

$$w(\tau)L + T(\tau) + i^{\text{Bonds}}(\tau)B(\tau) = P^B(\tau)\dot{B}(\tau) + \dot{M}^d(\tau) + P(t)c(\tau) \tag{29}$$

Equivalently, this constraint can be written as

$$\dot{v}(\tau) = \frac{w(\tau)}{P(\tau)} L + \frac{T(\tau)}{P(\tau)} + rv(\tau) - c(\tau) - [\pi + r]\, m(\tau) \tag{30}$$

where $m(\tau) = \frac{M(\tau)}{P(\tau)}$ are the household's real money holdings, $r$ is the real interest rate, $\pi$ is inflation and $v(\tau)$ is the household's real wealth, with $v(\tau) = m(\tau) + \frac{P^B(\tau)B(\tau)}{P(\tau)}$.[34] The resulting first-order conditions for the household's problem are:[35]

$$\frac{\theta}{1-\theta} \frac{c(\tau)}{m(\tau)} = r + \pi \tag{31}$$

and

$$[\eta + \theta\,(1-\eta)] \frac{\dot{c}(\tau)}{c(\tau)} - \theta(1-\eta) \frac{\dot{m}(\tau)}{m(\tau)} = r - \rho \tag{32}$$

Equation (31) is the familiar static efficiency requirement that in equilibrium the ratio of consumed goods' marginal utilities equal their cost ratio. The opportunity cost of money is shown to be equal to the sum of real interest rate $r$ and inflation rate $\pi$, i.e. the nominal interest rate.[36] The dynamic efficiency condition (32) is the standard version of the Keynes-Ramsey-Rule with money.[37] As usual, consumption growth depends positively on the difference between the real interest rate and the household's rate of time preference. In

[32] Endogenising labour supply by introducing labour into the utility function such that the consumption/leisure decision is not separable from the decision about real balances would of course add another channel through which money could have real effects, c.f. e.g. Walsh [2003], p. 66.

[33] $\dot{B}(\tau)$ is the change in the number of bonds held by the household.

[34] For a detailed derivation see Appendix 8.

[35] For a detailed derivation see Appendix 9.

[36] In our perfect-foresight model, there is no need to distinguish between expected and actual inflation, such that $i = r + \pi$.

[37] Cf., for example, Maussner [1994], p.177.

addition, it also depends positively (negatively) on the growth rate of real money balances if $u_{cm} > 0$ ($u_{cm} < 0$) where $u = \frac{(c(s)^{1-\theta} m(s)^{\theta})^{1-\eta} - 1}{1-\eta}$ is the household's instantaneous utility function.[38]

Since real interest rate $r$ and inflation $\pi$ are constant at steady state, equation (31) implies that the ratio of consumption to real balances held is constant. Thus, the growth rates of consumption and real balances must be equal. Using this in (32) gives the familiar form of the Euler equation:

$$\frac{\dot{c}(\tau)}{c(\tau)} = \frac{r-\rho}{\eta} \tag{33}$$

Equilibrium in the output market requires that the household's optimal choice of consumption equal production:

$$c(\tau) = Y(\tau) \tag{34}$$

# 3 General equilibrium

We will first use households' optimal behaviour and information from the public sector to determine equilibrium in the money market. We then introduce the compiled information into the production side equilibrium to analyse the model's general equilibrium.

## 3.1 Closing the model

### 3.1.1 Money market equilibrium

In the money market, money demand must equal supply, $M^s(\tau) = M^d(\tau)$ or given the initial money stock owned by households $M(0)$, the growth rate of real money supply, $\psi - \pi$,[39] must equal the growth rate of demand for real balances $\frac{\dot{m}^d(\tau)}{m^d(\tau)}$. From the household's problem, we have that $\frac{\dot{m}(\tau)}{m(\tau)} = \frac{\dot{c}(\tau)}{c(\tau)}$.[40] Further using output market equilibrium, we have that $\frac{\dot{c}(\tau)}{c(\tau)} = \gamma$, where $\gamma$ is the steady state growth rate of output.

Thus, equilibrium in the money market is characterised by $\psi - \pi = \gamma$ or

$$\pi = \psi - \gamma \tag{35}$$

that is, the inflation rate at steady state equals the difference between the money growth rate and the economy's output growth rate.

[38] $u_{cm} = \theta(1-\eta)(1-\theta) c^{-[\eta+\theta(1-\eta)]} m^{\theta(1-\eta)-1}$.

[39] Remember that the nominal supply is expanded at the constant rate $\frac{\dot{M}^s(\tau)}{M(\tau)} = \psi$.

[40] Cf. to the household's static optimality condition (31).

In Appendix 7 it is shown that the wage $w(\tau)$ grows at rate $\gamma+\pi$. Using (35) we thus have that the growth rate of marginal cost equals the growth rate of money supply:

$$\psi = \gamma + \pi = \omega \tag{36}$$

### 3.1.2 Research intensity in General Equilibrium

Using equilibrium in the money market (35), the equality of wage growth and money growth (36) and $i = r + \pi$, the zero profit condition (24) can be rewritten as

$$\frac{\frac{L}{\lambda} - \mu \frac{q^{\alpha-1}}{\alpha-1}}{\frac{\mu+\beta}{\mu+\beta-\alpha\psi}} \left[1 + \frac{\beta}{r+\mu+(\alpha-2)\gamma}\right] \left[r+\mu+\beta-2\gamma-\alpha(\psi-\gamma)\right]^{-1} = 1$$

Expected profit from research per invested Euro is given on the LHS of equation (24) and must equal the cost of investment (1 Euro) given on the RHS of the Equation. Expected profit is proportional to a new firm's value, which in line with the discussion in section 2.2.2, can be interpreted as the firm's discounted infinite stream of profits. The term $\left(\frac{L}{\lambda} - \mu\frac{q^{\alpha-1}}{\alpha-1}\right)\left(\frac{\mu+\beta}{\mu+\beta-\alpha\psi}\right)^{-1}$ corresponds to the firm's instantaneous profit, while the term $\left[1+\frac{\beta}{r+\mu+(\alpha-2)\gamma}\right]^{-1}\left[r+\mu+\beta-(\alpha(\psi-\gamma)-2\gamma)\right]$ represents the compound discount rate for the firm's future profits.[41] Rearranging, we have that the new firm's instantaneous profits must equal its compound discount rate:

$$\frac{\frac{L}{\lambda} - \mu \frac{q^{\alpha-1}}{\alpha-1}}{\frac{\mu+\beta}{\mu+\beta-\alpha\psi}} = \left[1 + \frac{\beta}{r+\mu+(\alpha-2)\gamma}\right]^{-1} \left[r+\mu+\beta-2\gamma-\alpha(\psi-\gamma)\right] \tag{37}$$

The two sides of equation (37) reflect the dependence of the optimal research intensity $\mu$ on the new firm's value. The LHS of equation (37) represents the firm's instantaneous profits, which rise with the size of the output sector $\left(\frac{L}{\lambda} - \mu\frac{q^{\alpha-1}}{\alpha-1}\right)/\left(\frac{\mu+\beta}{\mu+\beta-\alpha\psi}\right)$ which is proportional to total demand for the firm's good.[42]

Further using the Euler equation (33) and the equation relating economic growth to research intensity (21), we get an equation in $\mu$ and the model's parameters:

$$\frac{\frac{L}{\lambda} - \mu \frac{q^{\alpha-1}}{\alpha-1}}{\frac{\mu+\beta}{\mu+\beta-\alpha\psi}} = \left[1 + \frac{\beta}{\rho + \left(1+(\eta+\alpha-2)\frac{\alpha-1}{q^{\alpha-1}-1}\right)\mu}\right]^{-1} \left[\rho+\beta-\alpha\psi+\left(1+(\eta+\alpha-2)\frac{\alpha-1}{q^{\alpha-1}-1}\right)\mu\right] \tag{38}$$

[41] The discount factor consists of several parts because the firm's profit growth rate is different in periods with fixed prices than when prices can be changed. Cf. to section 2.2.2 for details.

[42] More precisely, the size of the final good sector is $\left[\frac{L}{\lambda} - \mu\frac{q^{\alpha-1}}{\alpha-1}\right]/\left\{\frac{\mu+\beta}{\mu+\beta-\alpha\omega}\left[\frac{P^*(\tau)}{P(\tau)Q(\tau)}\right]^{-\alpha}\right\}$. The term $\left[P^*(\tau)/\left(P(\tau)Q(\tau)\right)\right]^{-\alpha}$ cancels out, however, because the present value of the firm decreases in its chosen relative price relative to technology, $\left[P^*(\tau)/\left(P(\tau)Q(\tau)\right)\right]^{\alpha}$. Intuitively, relative price increases desired by all firms cancel out in equilibrium.

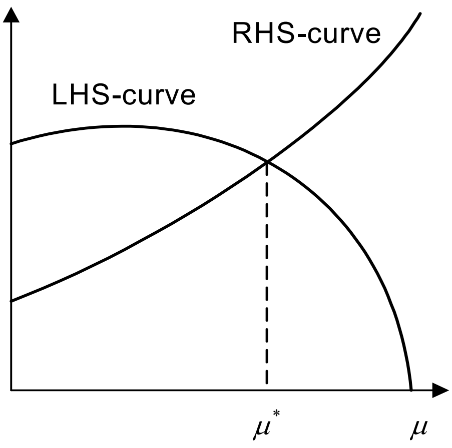


Figure 1: Equilibrium research intensity $\mu$

### 3.2 Existence and uniqueness of the steady state equilibrium

In this section we show that for sufficiently small frictions (large $\beta$) the economy has a unique steady state.[43] To do so, we first note that the RHS of equation (37) monotonously increases in $\mu$. Second we will show that the LHS of the equation is concave in $\mu$ as depicted in figure 1 and for sufficiently large $\beta$ has a bigger vertical intercept than the LHS-curve. Under this condition the economy has a unique steady state.

The RHS of equation (37) consists of the compound discount rate for the firm's profits. The higher the future research intensity $\mu$, the higher the probability that the new firm will quickly be replaced by a successor and thus the lower the probability that the firm will be active and accumulate profits in the future. This leads to an increase in the discount rate for future profits, reducing the present value of a given stream of profits. Therefore, the RHS of equation (37) increases in $\mu$.[44]

The LHS of equation (37) reflects a new firm's instantaneous profit which is proportional to the size of the output sector. The LHS is not monotonous in $\mu$ because there are several offsetting effects. First of all, the current level of $\mu$ has a negative effect on expected profit and thus the on LHS of equation (37) because an increase in $\mu$ raises demand for research labour $\mu\frac{q^{\alpha-1}}{\alpha-1}$ ('investment'), which given the size of the labour force is only compatible with labour market equilibrium for a smaller size of the output market ('consumption'). As demand for the new intermediate good is proportional to the size of

[43] If $\beta$ is too small, at most two steady states exist. The additional steady state is a by-product of excessive price stickiness. In our numerical calibrations condition 39 was compatible with our choices of $\beta$ based on empirical considerations, i.e. values of $\beta$ violating 39 lead to levels of prices rigidity exceeding empirically observed durations.

[44] When no rigidities are present in the model ($\beta = \infty$), the increase is linear in $\mu$, otherwise it is non-linear. In figures 2 and 2, the latter case is depicted.

the output sector,[45] it thus decreases in $\mu$, causing profits to drop. This effect is reflected in curve 1 in figure 2. At the same time, in the presence of price rigidity ($\beta < \infty$), the past level of $\mu$ has a positive effect on the size of the output sector because it reduces the distortion of demand caused by price rigidity reflected in the term $\left[(\mu+\beta)/(\mu+\beta-\alpha\psi)\right]^{-1}$:[46] For a given value of $\beta$, prices in the intermediate goods sector are changed the more often, the more innovations occur, and thus the higher $\mu$. Thus, aggregate price rigidity decreases in the past level of $\mu$ while the size of $Y(\tau)/Q(\tau)$ increases in the past level of $\mu$. Graphically, the term $\left[(\mu+\beta)/(\mu+\beta-\alpha\psi)\right]^{-1}$ pulls down curve 1, with the LHS-curve representing the product of the two effects. It is unclear which of the effects dominates, but the positive effect is the smaller, the higher $\beta$.[47] This is intuitive: A small value of $\beta$ implies a high degree of nominal rigidity (prices cannot be changed very often), entailing a very strong distortion of demand. Accordingly, relaxing the degree of rigidity by increasing $\mu$ has a strong positive effect on a new firm's value. When $\beta$ is big, prices are not very sticky. Therefore, relaxing the rigidity by increasing $\mu$ has got only a small positive effect on the firm's value compared to the negative effect of the increase in research labour. For $\beta \to \infty$, rigidity disappears and the LHS monotonously decreases in $\mu$. In figure 2, the LHS curve reduces to curve 1 in this case.

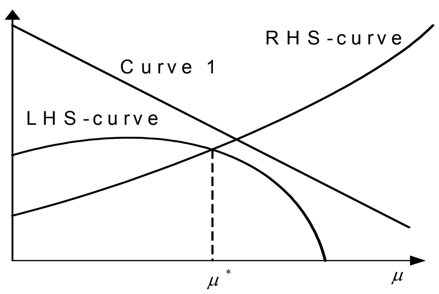


Figure 2:

For our purposes, it is sufficient that the LHS of equation (37) is strictly concave in $\mu$ for

[45] Cf. equation (3).

[46] The effect was first discussed in section 2.2.3 and will be analysed in more detail in section 4 where we present the comparative static effects of an increase in the money growth rate $\psi$.

[47] $\frac{\partial^2 LHS}{\partial\mu\partial\beta} = -\frac{\alpha\psi}{(\mu+\beta)^2}\left[\frac{q^{\alpha-1}}{\alpha-1} + \frac{2}{\mu+\beta}\left(\frac{L}{\lambda} - \mu\frac{q^{\alpha-1}}{\alpha-1}\right)\right] < 0.$

$\psi > 0$.[48] By further assuming that the conditions

$$\beta > \frac{1}{2}\left[-(\rho-\alpha\psi)+\left((\rho-\alpha\psi)^2+4\frac{\frac{L}{\lambda}\alpha\psi\rho}{\frac{L}{\lambda}-\rho}\right)^{1/2}\right] \tag{39}$$

$$\frac{L}{\lambda} > \rho \tag{40}$$

hold[49] we make sure that in equation (38) the following inequality holds:

$$\lim_{\mu\to 0}\left(\frac{L}{\lambda}-\mu\frac{q^{\alpha-1}}{\alpha-1}\right)\left(\frac{\mu+\beta}{\mu+\beta-\alpha\psi}\right)^{-1} > \lim_{\mu\to 0}\left[1+\frac{\beta}{\rho+\widetilde{\eta}\mu}\right]^{-1}\left[\rho+\beta-\alpha\psi+\widetilde{\eta}\mu\right]$$

where $\widetilde{\eta}=\left(1+(\eta+\alpha-2)\frac{\alpha-1}{q^{\alpha-1}-1}\right)$. Under these conditions, existence and uniqueness of the steady state equilibrium are ensured.[50]

# 4 Comparative statics

The comparative static properties of the steady state equilibrium will be analysed using the zero profit condition (37) and figures 1 and 3, where $LHS=\left(\frac{L}{\lambda}-\mu\frac{q^{\alpha-1}}{\alpha-1}\right)\left(\frac{\mu+\beta}{\mu+\beta-\alpha\psi}\right)^{-1}$ and
$RHS=\left[1+\frac{\beta}{r+\mu+(\alpha-2)\gamma}\right]^{-1}\left[r+\mu+\beta-2\gamma-\alpha\left(\psi-\gamma\right)\right]$.

**Economy without price rigidity: Superneutrality of money** First note that for the limiting case without rigidities ($\beta\to\infty$), the zero profit condition (37) reduces to

$$\frac{L}{\lambda}-\mu\frac{q^{\alpha-1}}{\alpha-1}=r+\mu+(\alpha-2)\gamma \tag{41}$$

which corresponds exactly to the case without money. All comparative static properties are those of the model without money. In particular, the research intensity is unaltered by both the level and growth rate of money supply. By equation (21) the economy's growth rate $\gamma$ is proportional to research intensity $\mu$. Hence, the growth rate of the economy is unaffected by monetary variables, too: In the quality ladder model with money in the utility function, money is neutral and superneutral if no additional frictions are introduced.

[48] $\frac{\partial^2 LHS}{\partial\mu^2}=-\frac{2\alpha\psi}{(\mu+\beta)^2}\left[\frac{q^{\alpha-1}}{\alpha-1}+\left(\frac{L}{\lambda}-\mu\frac{q^{\alpha-1}}{\alpha-1}\right)\frac{1}{\mu+\beta}\right]<0$ for $\psi>0$.
For $\psi=0$, $\frac{\partial LHS}{\partial\mu}<0$ and $\frac{\partial^2 LHS}{\partial\mu^2}=0$ Together with the condition (40) this is sufficient for the existence of a unique steady state in the case $\psi=0$, too.

[49] Equation (39) implies that a given level of rigidity $\beta$ entails an upper bound on the growth rate of money supply $\psi$ compatible with steady state equilibrium. The existence of an upper bound on money growth is not special to our model but applies to all standard NNS models as already acknowledged by, e.g. Ascari [2004] and King and Wolman [1996]. A condition similar to 40 is familiar from all specifications of the underlying real growth model, cf. e.g. Barro and Sala-i-Martin [2003].

[50] For a preliminary discussion of local stability cf. to section 6.

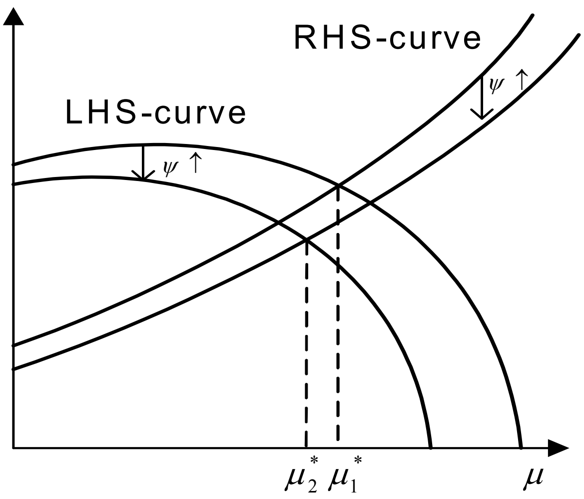


Figure 3: Comparative statics for $\beta < \infty$: money growth rate

**Economy with price rigidity: Two countervailing effects of money growth** We now use equation (37) to discuss in detail the comparative static properties of the steady state for $\beta < \infty$, with special regard to the effects of the growth rate of money supply, $\psi$, and the level of exogenous rigidity, $\beta$. We find that for $\beta < \infty$, the money growth rate $\psi$ has two clear-cut countervailing effects on economic growth which operate through money growth's influence on relative prices.

**Negative past relative price effect of money growth** As reflected in the term $[(\mu+\beta)/(\mu+\beta-\alpha\psi)]^{-1}$ on the LHS of equation (37), a positive rate of money growth allows for growth-reducing-effects of a given level of rigidity: For $\psi = 0$, rigidity is ineffective.[51] Firms optimally set their price as a markup over marginal cost, i.e. wages. As by equation (36) the rate of wage growth equals the growth rate of money supply, $\omega = \psi$, marginal cost is constant for $\psi = 0$ and thus, intermediate goods producers have no desire to readjust prices. The level of exogenous price rigidity $\beta$ is without importance. All firms charge the same relative price $P^*/P(\tau)$ and consequently, quality-adjusted demand, which depends on the relative price to the power of $-\alpha$, is equal for each intermediate good:[52]

$$\frac{x_{k_j}(\tau)}{q^{k_j(\alpha-1)}} = \left[\frac{P_{k_j}(\tau)}{P(\tau)}\right]^{-\alpha} Y(\tau) = \left[\frac{P^*}{P(\tau)}\right]^{-\alpha} Y(\tau)$$

Given that all quality-weighted intermediate goods have the same elasticity of substitution in the production of the final good, production is efficient.

A positive value of $\psi$ causes positive growth of marginal cost and of the optimal price. This means that because of past price rigidity there is a dispersion of prices between

[51] The equation for the equilibrium research intensity (37) reduces to the equation without rigidity, (41).

[52] Cf. to equation (3).

firms that have recently adjusted their price and those whose price has been fixed for a long time. Due to the rigidity, most intermediate goods are sold at prices set in the past, so they are cheap, given the level of technology, $Q(\tau)$. The average relative price is $\frac{P^*(\tau)}{P(\tau)}\left(\frac{\mu+\beta}{\mu+\beta-\alpha\psi}\right)^{-1/\alpha} < \frac{P^*(\tau)}{P(\tau)}$. Therefore, demand is distorted: cheap old intermediate goods are substituted for expensive new ones. Given the constant quality-weighted elasticity of substitution of intermediate goods in the production of the final good, this causes a deviation from the technically efficient mix of inputs. Consequently, the same amount of final goods is now produced with a higher input of intermediate goods.[53] This higher aggregate demand for intermediate goods translates into an increased demand for labour in the intermediate goods sector, which at given $\mu$ is only compatible with labour market equilibrium for a smaller size of the final good sector. Since demand for each intermediate good is proportional to the size of the final goods sector, price rigidity reduces demand for goods produced by new intermediate sector firms and therefore, their profits. The incentive to innovate is lower, causing a lower growth rate for the economy. As reflected in the term $[(\mu+\beta)/(\mu+\beta-\alpha\psi)]^{-1}$, this effect of exogenous and endogenous price rigidity ($\beta$ and $\mu$) on production efficiency and a new firm's profits is the more pronounced, the bigger $\psi$.[54] Thus, past price rigidity reduces economic growth. Graphically, an increase in $\psi$ causes a downward movement of the LHS curve in figure 3. Note that the effect of money growth is stronger, the bigger the level of rigidities present in the economy.[55]

**Positive future relative price effect of money growth** An increase in the money growth rate c.p. raises inflation $\pi = \psi - \gamma$. Given this growth rate of the price level, a new firm's relative price during periods of fixed prices erodes at rate $\pi$, which corresponds to an increased growth rate of demand $\alpha\pi$ for the good, i.e., higher profit growth.[56] The discount factor for profits in periods with fixed prices, $r+\mu+\beta-2\gamma-\alpha(\psi-\gamma)$, thus decreases. The fall in the discount rate increases the incentive to engage in research activities, raising economic growth.[57] In figure 3, this is reflected by the fact that an increase in $\psi$ causes a downward movement of the RHS curve, increasing the equilibrium research intensity. Analogously to the negative effect, the strength of the positive effect increases with the

[53] Cf. to equation (14) to see that aggregate demand for intermediate goods to produce a given level of output $Y(\tau)/Q(\tau)$ increases in the rigidity-caused term$(\mu+\beta)/(\mu+\beta-\alpha\psi)$.

[54] $\partial LHS/\partial\psi = -\alpha\left[L/\lambda - \mu q^{\alpha-1}/(\alpha-1)\right]/(\mu+\beta) < 0$.

[55] $\partial\left(\frac{\mu+\beta-\alpha\psi}{\mu+\beta}\right)/(\partial\psi\partial\beta) = \frac{\alpha}{(\mu+\beta)^2} > 0$.

[56] The effect of the diminishing price-cost ratio during intervals of fixed prices is compensated by the firm's corrected markup: The markup is chosen such that (given the development of marginal cost) the effect of the price on the present value of profits is zero (cf. to Appendix 3).

[57] $\partial RHS/\partial\psi = -\alpha\left[1+\frac{\beta}{r+\mu+(\alpha-2)\gamma}\right]^{-1} < 0$.

level of rigidity.[58]

The presence of potentially offsetting effects is consistent with the findings of empirical cross country studies which find small and often insignificant effects of money growth on economic growth.
In numerical calibrations, we found the net effect to be negative , cf. section 5.

**Influence of the exogenous level of rigidity** $\beta$ The level of exogenous rigidity $\beta$ influences the RHS of equation (37) via the both the discount rate applicable to periods where prices are sticky and the number of times prices can be reset. The firstmentioned discount rate increases in $\beta$ because the average duration of intervals where prices are sticky is shortened, which reduces the present value of profits. At the same time, the number of times prices can be reset rises, increasing profits. In our setting, the first effect dominates such that the compound discount rate increases in $\beta$, causing a lower growth rate. In figure 1, an increase in $\beta$ thus causes an upward shift of the RHS-curve.
This influence is counteracted by a lower level of rigidity-caused distortion: A higher value of $\beta$ means that prices can be changed more often, so demand is less distorted towards cheap old goods, reducing the inefficiency in the production of output and increasing the size of the output sector compatible with labour market equilibrium, as reflected in the term $[(\mu+\beta)/(\mu+\beta-\alpha\psi)]^{-1}$. Via the resulting increase in demand for the new good, an increase in $\beta$ raises the incentive to innovate and therefore, the growth rate. In figure 1, this is reflected by an upward shift of the LHS-curve.[59] Which of the two effects dominates is ambiguous. In numerical examples, the negative effect of rigidity dominates, cf. to section 5.

All other parameters have the standard influences on growth: As the household's rate of time preference $\rho$ increases, the interest rate $r$ rises, raising the discount rate for a new firm's profits. This has a negative effect on the equilibrium research intensity and growth rate, reflected by a downward movement of the RHS curve in figure 1. Further, as in the model without money, we have a scale effect of the size of the research-productivity-adjusted labour force $L/\lambda$ on the firm's profits and hence, the economy's growth rate.[60] In figure 1, an increase in $L/\lambda$ causes an upward movement of the LHS curve, thereby leading to a higher research intensity.

[58] $\partial\left(\frac{\mu+\beta-\alpha\psi}{\mu+\beta}\right)/(\partial\psi\partial\beta) > 0$.

[59] Note that this effect applies to the level of both exogenous rigidity $\beta$ and endogenous rigidity $\mu$: If prices can only be changed infrequently, innovation and the resulting price change act as a substitution for price flexibility, reducing the level of distortions in the economy.

[60] $\frac{\partial LHS}{\partial(L/\lambda)} = 1 - \frac{\alpha\psi}{\mu+\beta} > 0$.

## 5 Calibrating the model

We use standard parameter values from the literature and use steady state considerations to calibrate our model. Following standard NNS and RBC usage, the household's intertemporal rate of substitution $\eta^{-1}$ is assumed to be such that $\eta \in [0.5; 1]$. We choose $\alpha \in (6, 11)$ such that the markup in a steady state with constant marginal cost $\alpha/(\alpha-1)$ is between 10% and 20%, as is common in the literature. [61] The parameter $q$ indicates the size of the quality improvements made by innovating firms, we choose a 10% improvement in existing products. Based on empirical findings, prices are reported to be fixed for two to five quarters in the literature.[62] In our model, the average interval of fixed prices is partly endogenous and given by $(\beta+\mu)^{-1} = \left(\beta + \frac{\alpha-1}{q^{\alpha-1}-1}\gamma\right)^{-1}$. Based on our choice of $q$ and the range of values for $\alpha$, we choose $\beta \in (1, 2)$. Given that $\gamma$ is nonnegative, this entails an upper bound on price rigidity of one year when growth is at its minimum. Further, the household's discount rate is $\rho = 0.035$. We then calibrate $L/\lambda$ such that in the steady state with constant marginal cost, the growth rate is $\gamma = 0.02$.
For our baseline case, we have chosen the following constellation:

| Parameter | value | Parameter | value |
|---|---|---|---|
| $\eta$ | 2 | $\beta$ | 1.25 |
| $\alpha$ | 8 | $q$ | 1.1 |
| $\rho$ | 0.035 | $L/\lambda$ | 0.38365 |

At this parameter constellation, we have that the growth rate $\gamma$ is a decreasing function of the money supply growth rate $\psi$, the rigidity-induced distortion effect clearly dominates the individual relative price effect.

In line with the empirical literature, the net effect of an increase in the money growth rate is small: Increasing the annual money growth rate from 1% to 10% reduces economic growth from 1.99% to 1.73% p.a.[63]

Individually, the two effects are much more sizeable: Holding constant the positive effect of money growth on economic growth, the negative effect of an increase of $\psi$ from

[61] Cf. e.g. to Chari, Kehoe and McGrattan [200]. Markups of this dimension are reported for the US by Basu and Fernald [1995, 1997] and for Germany by Linnemann [1999].

[62] Taylor [2000] summarises the empirical literature as indicating one year as the average frequency of price changes. Based on these empirical findings, prices are assumed to be fixed for one year in the baseline calibrations of many DSGE models such as Chari, Kehoe and McGrattan [2000]. Yet recent empirical research finds shorter periods of rigid prices: Blinder et al [1998 ] indicate an average time between price changes of nine months for US firms while the result of Bils and Klenow [2004] is under six months.

[63] For example, Barro [1996] reports that a 10 percentage point increase in inflation (M1 growth) reduces annual GDP growth by about 0.24 (0.23) percentage points in a panel regression covering data from over 100 countries for the period 1960-1990. Fischer [1993] reports results of a similar magnitude.

1% to 10% reduces the growth rate by some 1.23 percentage points to 0.76%. At the same time, holding constant the negative effect, the positive effect of the increase in $\psi$ on the growth rate amounts to 1.30 percentage points (the growth rate would be 3.29%).

The net effect of rigidity on growth is negative: Decreasing $\beta$ from $\beta = 1.25$ to $\beta = 0.75$ at a given money growth rate reduces economic growth slightly (For $\psi = 0.01$, the growth rate drops from 1.986% to 1.983%). The effect of rigidity on growth is stronger for higher rates of money growth and vice versa. For $\psi = 0.1$, the growth rate drops from 1.732% to 1.669% when $\beta$ is decreased from $\beta = 1.25$ to $\beta = 1.15$. The drop in the growth rate resulting from an increase of $\psi$ from 1% to 10% growth is -.31 percentage points at $\beta = 1.15$, compared to -.26 percentage points at $\beta = 1.25$.

Varying the underlying parameters in the ranges indicated above did not bring about qualitative changes in the results.

# 6 Stability

So far, the system has been investigated for its local stability properties at the steady states corresponding to the cases without price rigidity ($\beta \to \infty$) and the case with maximum rigidity ($\beta = 0$). For this purpose, the model's variables have been transformed such that they are constant at steady state. The first order conditions have then been approximated around the respective steady state using a linear Taylor approximation.

For $\beta \to \infty$, the system is 2-dimensional in the variables $\frac{Y}{Q}$ and $\frac{m}{Q} = \frac{M}{PQ}$ which are both jump variables when prices are flexible.[64] We find that both eigenvalues of the system are positive such that the system is saddle-path stable.[65]

For $\beta = 0$, the system consists of the variables $\frac{Y}{Q}$, $\frac{m}{Q}$, $\mu$, $\pi$ and $\widetilde{w} = \frac{w}{PQ}$ where the variable reflecting real money holdings $\frac{m}{Q}$ is now predetermined. Due to the system's dimension, the stability properties cannot be analysed analytically. Starting from a baseline calibration and then varying parameters, we checked some numerical examples and found the relevant steady state to be locally stable but indeterminate.[66]

This implies that the system has some Keynesian features that might be of interest for the joint analysis of growth and business cycles. We aim to address this question in the

[64] For simplicity, we assume $\eta = 1$ (the household's utility is logarithmic in the consumption bundle $c(s)^{1-\theta}m(s)^{\theta}$).

[65] Note that for systems that only contain jump variables, the constellation with all eigenvalues being positive is the equivalent of saddle path stability, although the stable arm is of dimension zero. Cf. to Benhabib, Schmitt-Grohé and Uribe [2001]) for details.

[66] Note that for $\beta = 0$ there exists a second steady state associated with a smaller value of the research intensity (Cf. footnote 43). We found this steady state to be locally stable and determinate in the majority of cases, while it was stable but indeterminate in eight examined cases.

intermediate cases ($0 < \beta < \infty$).

# 7 Conclusion

We find that the Schumpeterian growth model with small nominal frictions has a unique steady state. In this steady state, money growth determines the growth rate of prices optimally charged by intermediate goods firms. Price rigidity leads to disparity in the prices charged by firms and thereby to two countervailing effects of money growth on productivity growth: Demand is distorted towards cheap 'old' intermediate goods', which leads to inefficient output production and a reduction in the demand for new intermediate goods, lowering the incentive to innovate. Yet inflation erodes the relative price of a new intermediate good in periods where its price is fixed, boosting demand for the good and leading thus to a rise in the incentive to innovate. Both effects of rigidity increase in money growth. In our calibrations, the negative effect of money growth and price rigidity on economic growth dominates, with moderate inflation rates causing a sizeable reduction in the rate of economic growth.

From a normative point of view, growth may be both higher or lower than optimal in the underlying frictionless Aghion-Howitt model. A natural next step will be to analyse the welfare implications of inflation in the present model: Does inflation increase or reduce the wedge between the social optimal growth rate and the decentral growth rate?

# 8 Appendix

**Appendix 1** ***Final goods sector cost minimization problem***

The firm's problem is to chose intermediate good quantities $x_{k_j}(\tau)$ to minimise the production cost for a given amount of output produced according to the production function (1):

$$\min_{\left\{x_{k_j}\right\}} L = \sum_{j=1}^{N} \left(P_{k_j}(\tau) x_{k_j}(\tau)\right) + P(\tau) \left\{ Y(\tau) - \left[ \sum_{j=1}^{N} \left( q^{k_j(\tau)} x_{k_j}(\tau) \right)^{\frac{\alpha-1}{\alpha}} \right]^{\frac{\alpha}{\alpha-1}} \right\}$$

The Lagrange multiplier on the quantity constraint is the output price $P$ as the multiplier has the interpretation of the marginal cost of one additional unit of output, which is precisely given by the price index. The corresponding first order conditions are

$$P_{k_j}(\tau) = P(\tau) \frac{\alpha}{\alpha-1} \left[ \sum_{i=1}^{N} \left( q^{k_i(\tau)} x_{k_i}(\tau) \right)^{\frac{\alpha-1}{\alpha}} \right]^{\frac{\alpha}{\alpha-1}-1} \frac{\alpha-1}{\alpha} \left( q^{k_j(\tau)} x_{k_j}(\tau) \right)^{-\frac{1}{\alpha}} q^{k_j(\tau)} \quad (42)$$

$$Y(\tau) = \left[ \sum_{j=1}^{N} \left( q^{k_j(\tau)} x_{k_j}(\tau) \right)^{\frac{\alpha-1}{\alpha}} \right]^{\frac{\alpha}{\alpha-1}} \tag{43}$$

when $L$ is maximised with respect to $x_{k_j}(\tau)$ and the Lagrange multiplier $P(\tau)$, respectively. Using (43) in (42) and solving for $x_{k_j}(\tau)$ leads to equation (3) in the main text.

The form of the price index $P(\tau)$ can be derived as follows: Multiplying condition (42) with $x_{k_j}(\tau) > 0$ and aggregating over intermediate products $j$ results in

$$\sum_{j=1}^{N} P_{k_j}(\tau) x_{k_j}(\tau) - P(\tau) \left[ \sum_{i=1}^{N} \left( q^{k_i(\tau)} x_{k_i}(\tau) \right)^{\frac{\alpha-1}{\alpha}} \right]^{\frac{\alpha}{\alpha-1}-1} \sum_{j=1}^{N} \left( q^{k_j(\tau)} x_{k_j}(\tau) \right)^{\frac{\alpha-1}{\alpha}} = 0$$

Solving for $P(\tau)$ and using (43) entails

$$P(\tau)Y(\tau) = \sum_{j=1}^{N} \left( P_{k_j}(\tau) x_{k_j}(\tau) \right) \tag{44}$$

Using the equation (3) specifying optimal demand for good $x_{k_j}$ in equation (44) leads to

$$P(\tau)Y(\tau) = \sum_{j=1}^{N} \left( P_{k_j}(\tau) \left( \frac{P_{k_j}}{P(\tau)} \right)^{-\alpha} q^{k_j(\alpha-1)} Y(\tau) \right)$$

which can be rearranged to yield equation (4) in the main text. Note that this price corresponds to the minimum cost of obtaining one unit of $Y(\tau)$.

## Appendix 2 *Probabilities for an intermediate good producing firm's pricing problem*

When entering the market, a firm in sector $j$ sets a price to maximise the discounted sum of profits $G_{k_j}(s)$ for each future period $s$ weighted by the probability $\Pr(s)$ that the firm has not been replaced and has not received a signal to reset its price by period $s$. As the two events 'reset signal' and 'innovation' are governed by two distinct Poisson processes, they are stochastically independent such that we can express the joint probability as the product of the individual probabilities.

$$E\left[ PV_{k_j} \middle| t_{k_j} \right] = \int_{t_{k_j}}^{\infty} e^{-i(s-\tau)} G_{k_j}(s) \Pr(s)\, ds$$

$D(\tau)$ is the cumulative density function for the probability that the date of the next innovation in sector $j$, $t_{k_j+1}$, will precede time $\tau$, given that the last innovation was generated at time $t_{k_j}$: $D(\tau) = p(t_{k_j+1} \leq \tau)$. Then the probability of not having been replaced by another firm by time $\tau$ is $1 - D(\tau)$. We find the function $D(\tau)$ by first noticing that its time derivative, $\frac{dD(\tau)}{d\tau}$, equals the flow probability of an innovation occurring at time $\tau$, which is the parameter of the homogeneous Poisson process, $\mu_{k_j}$, given that no innovation was generated between $t_{k_j}$ and $\tau$:

$$\frac{dD(\tau)}{d\tau} = (1 - D(\tau))\,\mu_{k_j} \tag{45}$$

We then define $\widetilde{D}(\tau)$ as the probability density of an innovation being generated after time $\tau$: $\widetilde{D}(\tau) = 1 - D(\tau)$, which implies $\frac{d\widetilde{D}(\tau)}{d\tau} = -\frac{dD(\tau)}{d\tau}$. Using (45) gives $\frac{d\widetilde{D}(\tau)}{d\tau} + \widetilde{D}(\tau)\mu_{k_j} = 0$. This first-order linear differential equation has the solution

$$\widetilde{D}(\tau) = Ae^{-\mu_{k_j}(\tau - t_{k_j})} \tag{46}$$

where $A$ is a constant. The probability of two innovations occurring at the same time is negligible, so $D(t_{k_j}) = 0$ which implies $\widetilde{D}(t_{k_j}) = 1$. Using the fact that $t_{k_j+1} \geq t_{k_j}$ and the boundary condition in (46) yields $A = 1$ and the solution

$$\widetilde{D}(\tau) = 1 - D(\tau) = e^{-\mu_{k_j}(\tau - t_{k_j})}$$

Analogously, given a Poisson process with parameter $\beta$, the probability of not receiving a reset signal between times $\tau$ and $s$ is $1 - B(\tau) = \widetilde{B}e^{-\beta(\tau - t_{k_j})}$. Since we have no information about the time the last reset signal was received in the sector, we cannot definitize the constant $\widetilde{B}$. Thus, the joint probability that the firm has not been replaced and has not received a pricing signal by time $\tau$ given that the last innovation was at time $t_{k_j}$ is $\widetilde{B}e^{-\left(\mu_{k_j}+\beta\right)(\tau - t_{k_j})}$which leads to equation (7) in the text.

## Appendix 3 *Solution of an intermediate good producing firm's optimal pricing problem*

The firm chooses a price $P_{k_j}(\tau)$ to maximise the present value of profits when entering the market, given by (8). Using stationary growth of output $Y$, output price $P$ and wage $w$ with rates $\gamma$, $\pi$ and $\omega$, respectively, the firm solves

$$\underset{P_{k_j}(\tau)}{max}\, E\left[PV\left|t_{k_j} = \tau\right.\right] = q^{k_j(\alpha-1)}P(\tau)^{\alpha}Y(\tau)\int\limits_{\tau}^{\infty} e^{-(i+\mu+\beta-\alpha\pi-\gamma)(s-\tau)}\left[\left[P_{k_j}(\tau)\right]^{1-\alpha} - w(s)P_{k_j}(\tau)^{-\alpha}\right]ds.$$

The first order condition for a maximum is

$$(1-\alpha)\left[P_{k_j}(\tau)\right]^{-\alpha}\int\limits_{\tau}^{\infty} e^{-(i+\mu+\beta-\alpha\pi-\gamma)(s-\tau)}ds + w(\tau)\alpha P_{k_j}(\tau)^{-\alpha-1}\int\limits_{\tau}^{\infty} e^{-(i+\mu+\beta-\alpha\pi-\gamma-\omega)(s-\tau)}ds = 0$$

Solving the integral and rearranging yields equation (9). It is trivial to show that the second order condition for a maximum holds.

## Appendix 4 *A new intermediate good producing firm's market value*

The firm's market value is the discounted sum of profits from future periods $s$ where the profits are weighted due to two independent sources of uncertainty: The first weight is given by the probability $e^{-\mu(s-\tau)}$ of not having been replaced by time $s$. The second source of uncertainty is given by the firm's price in period $s$: The price charged can be any $P^*(\theta)$ with $\theta \in (\tau, s)$ depending on whether any reset signal for the price was received between $\tau$ and $s$ and when it was received. Thus, the price charged at time $s$ can be represented as a weighted sum of the past optimal prices, where the weights are as follows: The flow probability that a signal to reset prices was received in period $\theta$ is $\beta$. With probability $e^{-\beta(s-\theta)}$, no signal was received between $\theta$ and $s$.[67] As these two events are independent, the probability of having last reset one's price due to a pricing signal at $\theta \in (\tau, s)$ is $\beta e^{-\beta(s-\theta)}$. Additionally, if no reset signal has been received up to period $s$, the firm's price will continue to be $P^*(\tau)$, which has probability $\left(1 - \int_\tau^s \beta e^{-\beta(s-\theta)} d\theta\right)$. Since the processes for innovations and reset signals are independent, the joint probability of the described events takes on the multiplicative form reflected in equation (11).

Making use of the fact that the optimal price grows with the growth rate of marginal cost, $\omega$, and solving the integrals associated with the price at time $s$ yields

$$\begin{aligned} E(V_j|\tau) &= A(\tau) \int_\tau^\infty e^{-\chi(s-\tau)} P^*(s)^{1-\alpha} \left[\beta \frac{1-e^{-[\beta-(\alpha-1)\omega](s-\tau)}}{\beta-(\alpha-1)\omega} + e^{-[\beta-(\alpha-1)\omega](s-\tau)}\right] ds \\ &\quad - A(\tau) w(\tau) \int_\tau^\infty e^{-(\chi-\omega)(s-\tau)} P^*(s)^{-\alpha} \left[\beta \frac{1-e^{-(\beta-\alpha\omega)(s-\tau)}}{\beta-\alpha\omega} + e^{-(\beta-\alpha\omega)(s-\tau)}\right] ds \end{aligned}$$

where we define $\chi = i + \mu - \alpha\pi - \gamma$ and $A(\tau) = q^{k_j(\alpha-1)} Y(\tau) P(\tau)^\alpha$. Using again constant growth of $P^*$ and rearranging terms, we have

$$\begin{aligned} E(V_j|\tau) &= A(\tau) \frac{P^*(\tau)^{1-\alpha}}{\beta-(\alpha-1)\omega} \left[\beta \int_\tau^\infty e^{-[\chi-(1-\alpha)\omega](s-\tau)} ds - (\alpha-1)\omega \int_\tau^\infty e^{-(\chi+\beta)(s-\tau)} ds\right] \\ &\quad - A(\tau) w(\tau) \frac{P^*(\tau)^{-\alpha}}{\beta-\alpha\omega} \left[\beta \int_\tau^\infty e^{-[\chi-(1-\alpha)\omega](s-\tau)} ds - \alpha\omega \int_\tau^\infty e^{-[\chi+\beta-\omega](s-\tau)} ds\right] \end{aligned}$$

Solving the remaining integrals, we have

$$E(V_j|\tau) = \frac{A(\tau)}{P^*(\tau)^\alpha} \left\{ \frac{P^*(\tau)}{\beta-(\alpha-1)\omega} \left[\frac{\beta}{\chi-(1-\alpha)\omega} - \frac{(\alpha-1)\omega}{\chi+\beta}\right] - \frac{w(\tau)}{\beta-\alpha\omega} \left[\frac{\beta}{\chi-(1-\alpha)\omega} - \frac{\alpha\omega}{\chi+\beta-\omega}\right] \right\}$$

Expanding the fractions inside the brackets, we have

$$E(V_j|\tau) = A(\tau) P^*(\tau)^{-\alpha} \frac{\chi+\beta+(\alpha-1)\omega}{\chi-(1-\alpha)\omega} \left[P^*(\tau) \frac{1}{\chi+\beta} - w(\tau) \frac{1}{\chi+\beta-\omega}\right]$$

[67] Here, we have been able to definitize the constant $\widetilde{B} = 1$ since we know that the probability of receiving two or more signals at time $\theta$ is negligible, such that $B(\theta) = 0$.

Finally using the equation for the optimal price (9) and reinserting $\chi = i + \mu - \alpha\pi - \gamma$ and $A(\tau) = q^{k_j(\alpha-1)} Y(\tau) P(\tau)^{\alpha}$ we have equation (12) in the main text.

### Appendix 5 *The Intermediate good sector's labour demand*

Equation (13) can be rewritten as

$$L^X(\tau) = Y(\tau)P(\tau)^{\alpha} \sum_{k=1}^{k^{\max}} d_k(\tau)\, q^{k(\alpha-1)} \sum_{k_j=k} \left(P_{k_j}(\tau)\right)^{-\alpha}$$

where $d_k(\tau)$ is the number of sectors at quality rung $k$ at time $\tau$

Following (Benhabib, Schmitt-Grohé and Uribe [2001a] and [2001b], Leith andWren-Lewis [2000] and Wolman [1999]), the average price effective at time $\tau$ can be expressed as a weighted average of past optimal prices, where the weights $\widetilde{f}(s,\tau)$ refer to the probability that a price valid at time $\tau$ has not been changed since time $s$. This implies

$$\sum_{j=1}^{N} P_{k_j}(\tau)^{-\alpha} = \int_{-\infty}^{\tau} \widetilde{f}(s,\tau)\,[P^*(s)]^{-\alpha}\, ds$$

As the timing of innovations is independent of a sector's position on the quality ladder $q^{k_j}$, the structure of prices for a given $q^k$ is the same as the structure for all sectors. Thus we have

$$\sum_{k_j=k} P_{k_j}(\tau)^{-\alpha} = \sum_{j=1}^{N} P_{k_j}(\tau)^{-\alpha} = \int_{-\infty}^{\tau} \widetilde{f}(s,\tau)\,[P^*(s)]^{-\alpha}\, ds$$

and

$$\begin{aligned} L^X(\tau) &= P(\tau)^{\alpha} Y(\tau) \sum_{k=1}^{k^{\max}} d_k(\tau)\, q^{k(\alpha-1)} \int_{-\infty}^{\tau} \widetilde{f}(s,\tau)\, P^*(s)^{-\alpha} ds \\ &= P(\tau)^{\alpha} Q(\tau)^{\alpha} \frac{Y(\tau)}{Q(\tau)} \int_{-\infty}^{\tau} \widetilde{f}(s,\tau)\, P^*(s)^{-\alpha} ds \end{aligned} \tag{47}$$

A price in effect at time $\tau$ dates from time $s$ if there was either an innovation *or* a pricing signal at time $s$ *and* if there was no innovation between times $s$ and $\tau$ *and* if no pricing signal was received in the same period. As explained in appendix 2, Innovations and pricing signals are governed by two independent Poisson processes with parameters $\mu$ and $\beta$, respectively. Thus, the probability of an innovation (not) occurring is independent from the probability of a pricing signal (not) being received. Therefore, the probability of no innovation *and* no pricing signal between times $s$ and $\tau$ is $e^{-\mu(\tau-s)}e^{-\beta(\tau-s)} = e^{-(\mu+\beta)(\tau-s)}$. The flow probability of an innovation *or* a pricing signal occurring at time $s$ is $\mu+\beta$. Thus, we have $\widetilde{f}(s,\tau) = (\mu+\beta)\, e^{-(\mu+\beta)(\tau-s)}$. Using this and steady growth of $P^*$ at rate $\omega$ in equation (47), we have

$$L^X(\tau) = \left[\frac{P^*(\tau)}{P(\tau)Q(\tau)}\right]^{-\alpha} \frac{Y(\tau)}{Q(\tau)} \int_{-\infty}^{\tau} (\mu+\beta)\, e^{-(\mu+\beta)(\tau-s)} e^{(\alpha\omega)(\tau-s)} ds$$

Solving the integral which converges for $\mu+\beta>\alpha\omega$ leads to equation () in the main text.

## Appendix 6 *Development of the aggregate Quality index*

The expected growth rate of the aggregate quality index is

$$E\left[\widehat{Q(\tau)}\right]=E\left\{\left[\widehat{\sum_{j=1}^{N}q^{(\alpha-1)k_j(\tau)}}\right]^{\frac{1}{\alpha-1}}\right\}=\frac{1}{\alpha-1}E\left[\widehat{\sum_{j=1}^{N}q^{(\alpha-1)k_j(\tau)}}\right] \tag{48}$$

where a hat denotes the proportional growth rate of variable $x$, $\widehat{x}=\frac{dx/dt}{x}$. The growth rate of the sum in equation (48) is equal to the sum of the individual growth rates weighted with the sector's share in the aggregate quality index[68]:

$$E\left[\widehat{Q(\tau)}\right]=\frac{1}{\alpha-1}\sum_{j=1}^{N}E\left[\widehat{q^{(\alpha-1)k_j(\tau)}}\frac{q^{(\alpha-1)k_j(\tau)}}{\sum_{j=1}^{N}q^{(\alpha-1)k_j(\tau)}}\right]$$

The expected value of the product of two random variables is given by $E[XY]=E[X]\,E[Y]+Cov[XY]$, where $Cov[XY]$ denotes the covariance between $X$ and $Y$. Thus, we have $\Rightarrow$

$$E\left[\widehat{Q(\tau)}\right]=\frac{1}{\alpha-1}\sum_{j=1}^{N}\left\{E\left[\widehat{q^{(\alpha-1)k_j(\tau)}}\right]E\left[\frac{q^{(\alpha-1)k_j(\tau)}}{\sum_{j=1}^{N}q^{(\alpha-1)k_j(\tau)}}\right]+Cov\left[\widehat{q^{(\alpha-1)k_j(\tau)}}\frac{q^{(\alpha-1)k_j(\tau)}}{\sum_{j=1}^{N}q^{(\alpha-1)k_j(\tau)}}\right]\right\}$$

Using (19),

$$E\left[\widehat{Q(\tau)}\right]=\sum_{j=1}^{N}\mu(\tau)\frac{q^{\alpha-1}-1}{\alpha-1}E\left[\frac{q^{(\alpha-1)k_j(\tau)}}{\sum_{j=1}^{N}q^{(\alpha-1)k_j(\tau)}}\right]+\frac{1}{\alpha-1}\sum_{j=1}^{N}Cov\left[\widehat{q^{(\alpha-1)k_j(\tau)}}\frac{q^{(\alpha-1)k_j(\tau)}}{\sum_{j=1}^{N}q^{(\alpha-1)k_j(\tau)}}\right]$$

Using the definition of the aggregate Quality index (5) results in

$$E\left[\widehat{Q(\tau)}\right]=\mu\frac{q^{\alpha-1}-1}{\alpha-1}+\frac{1}{\alpha-1}\sum_{j=1}^{N}Cov\left[\widehat{q^{(\alpha-1)k_j(\tau)}}\frac{q^{(\alpha-1)k_j(\tau)}}{\sum_{j=1}^{N}q^{(\alpha-1)k_j(\tau)}}\right]$$

As was shown in section 2.3, the probability of an innovation being made and thus $\widehat{q^{(\alpha-1)k_j(\tau)}}$ is independent of sector $j$'s position relative to the other sectors, $\frac{q^{(\alpha-1)k_j(\tau)}}{\sum_{j=1}^{N}q^{(\alpha-1)k_j(\tau)}}$. Thus, $Cov\left[\widehat{q^{(\alpha-1)k_j(\tau)}}\frac{q^{(\alpha-1)k_j(\tau)}}{\sum_{j=1}^{N}q^{(\alpha-1)k_j(\tau)}}\right]=0$.

Consequently,

$$E\left[\widehat{Q(\tau)}\right]=\mu\frac{q^{\alpha-1}-1}{\alpha-1}$$

The Law of large numbers implies that for a large number $N$ of sectors, the actual growth rate of the quality index, $\gamma_Q(\tau)$, converges to the expected growth rate, $E\left[\widehat{Q(\tau)}\right]$.

[68] For example, $\widehat{X+Y}=\frac{\dot{(X+Y)}}{X+Y}=\widehat{X}\frac{X}{X+Y}+\widehat{Y}\frac{Y}{X+Y}$.

More precisely, a standard version of the law of large numbers for independent and identically distributed variables $\{Y_t\}$ with $E[Y_t] = \mu$ and variance $E\left[(Y_t - \mu)^2\right] = \sigma^2$ states that the sample mean $\overline{Y}_T = (1/T)\sum_{t=1}^{T} Y_t$ converges in probability to the population mean $\mu$. $\overline{Y}_T$ has expectation $\mu$ and variance $E\left[\left(\overline{Y}_T - \mu\right)^2\right] = \sigma^2/T$ . This variance goes to zero as $T \to \infty$ implying $\overline{Y}_T \xrightarrow{p} \mu$. Cf. e.g. Hamilton [1994]. In the present context, the weighting factors $(q^{(\alpha-1)k_j(\tau)}/Q(\tau)^{(\alpha-1)})$ are not constant as in the standard case $(1/T)$. Yet their being small and summing up to unity suffices for the result to carry over to our case. Thus, we have equation (20) in the main text.

## Appendix 7 *The growth rate of wages at steady state*

Analogously to the procedure in appendix 5, the output price (4) at steady state can be rewritten as

$$P(\tau)^{1-\alpha} = Q(\tau)^{(\alpha-1)}\,(\mu+\beta)\int_{-\infty}^{\tau} e^{-(\mu+\beta)(\tau-s)}P^*(s)^{1-\alpha}ds$$

>From (9), we have that the optimal price at steady state grows at rate $\omega$, such that

$$P(\tau)^{1-\alpha} = Q(\tau)^{(\alpha-1)}\,(\mu+\beta)\,P^*(\tau)^{1-\alpha}\int_{-\infty}^{\tau} e^{-[\mu+\beta-(\alpha-1)\omega](\tau-s)}ds$$

Calculating the integral's value and rearranging terms leads to

$$\left[\frac{P^*(\tau)}{P(\tau)Q(\tau)}\right]^{\alpha-1} = \frac{\mu+\beta}{\mu+\beta-(\alpha-1)\,\omega} \tag{49}$$

where we require $\mu+\beta-(\alpha-1)\,\omega > 0$ for the integral to converge.

As the RHS of (49) is constant, so must be the left hand side of the equation. Thus, with $P^*(\tau) = \frac{\alpha}{\alpha-1}\frac{i+\mu+\beta-\alpha\pi-\gamma}{i+\mu+\beta-\alpha\pi-\gamma-\omega}w(\tau)$, we have $\frac{\dot{w}(\tau)}{w(\tau)} = \frac{\dot{P}(\tau)}{P(\tau)} + \frac{\dot{Q}(\tau)}{Q(\tau)}$ or

$$\omega = \pi + \gamma$$

The wage growth rate at the steady state equals the sum of inflation and growth rate, where we have used that $\gamma_Q = \gamma$ at the steady state.

## Appendix 8 *Household's budget constraint*

In real terms, the household's budget constraint (29) can be written as

$$\frac{w(\tau)}{P(\tau)}L + \frac{T(\tau)}{P(\tau)} + \left[i(\tau) - \frac{\dot{P}^B(\tau)}{P^B(\tau)}\right]\frac{P^B(\tau)B(\tau)}{P(\tau)} = \frac{P^B(\tau)}{P(\tau)}\dot{B}(\tau) + \frac{\dot{M}^d(\tau)}{M^d(\tau)}m(\tau) + c(\tau)$$

where we have used equation (27) to replace $i^{\text{Bonds}}(\tau)$. The change in the household's real money holdings is $\dot{m}(\tau) = \frac{\dot{M^d}(\tau)P(\tau) - \dot{P}(\tau)M^d(\tau)}{P(\tau)^2} = m(\tau)\left[\frac{\dot{M^d}(\tau)}{M^d(\tau)} - \pi(\tau)\right]$, where $\pi(\tau) = \frac{\dot{P}(\tau)}{P(\tau)}$ is the inflation rate. Using this and rearranging terms yields

$$\frac{w(\tau)}{P(\tau)}L + \frac{T(\tau)}{P(\tau)} + i(\tau)\frac{P^B(\tau)B(\tau)}{P(\tau)} = \frac{\dot{P^B}(\tau)}{P^B(\tau)}\frac{P^B(\tau)B(\tau)}{P(\tau)} + \frac{\dot{B}(\tau)}{B(\tau)}\frac{P^B(\tau)B(\tau)}{P(\tau)} + \dot{m}(\tau) + \pi(\tau)\,m(\tau) + c(\tau)$$

Using the definition of the nominal interest rate, $i = r + \pi$ and collecting terms, we have

$$\frac{w(\tau)}{P(\tau)}L + \frac{T(\tau)}{P(\tau)} + r(\tau)\frac{P^B(\tau)B(\tau)}{P(\tau)} = \left[\frac{\dot{P^B}(\tau)}{P^B(\tau)} + \frac{\dot{B}(\tau)}{B(\tau)} - \pi(\tau)\right]\frac{P^B(\tau)B(\tau)}{P(\tau)} + \dot{m}(\tau) + \pi(\tau)m(\tau) + c(\tau)$$

The household's nominal wealth $V(\tau)$ consists of his money and bond holdings, $V(\tau) = M(\tau) + P^B(\tau)B(\tau)$. Equivalently, real wealth is $v(\tau) = \frac{V(\tau)}{P(\tau)} = m(\tau) + \frac{P^B(\tau)B(\tau)}{P(\tau)}$. The instantaneous change in the household's real wealth is

$$\dot{v}(\tau) = \dot{m}(\tau) + \left[\frac{P^B(\tau)B(\tau)}{P(\tau)}\right]^{\cdot} = \dot{m}(\tau) + \left[\frac{\dot{P^B}(\tau)}{P^B(\tau)} + \frac{\dot{B}(\tau)}{B(\tau)} - \frac{\dot{P}(\tau)}{P(\tau)}\right]\frac{P^B(\tau)B(\tau)}{P(\tau)} \tag{50}$$

Using this, the budget constraint can be rewritten as in equation (30).

## Appendix 9 *Household's optimization problem*

We define $z(\tau)$ as the fraction of the household's real wealth invested in bonds: $z(\tau) = \frac{\frac{P^B(\tau)B(\tau)}{P(\tau)}}{v(\tau)}$ and rewrite utility function and budget constraint (50) accordingly. The household's problem consists in choosing optimal paths for consumption $\{c\}_{\tau=0}^{\infty}$ and the investment strategy $\{z\}_{\tau=0}^{\infty}$ to maximise the utility function (28) subject to the budget constraint (30). We set up the current-value Hamiltonian

$$H = \frac{\left\{c^{1-\theta}\left[(1-z)\,v\right]^{\theta}\right\}^{1-\eta} - 1}{1-\eta} + \xi\left[\frac{w}{P}L + T^{real} + rzv - c - \pi\,(1-z)\,v\right]$$

where we have used $z(\tau)v(\tau) = \frac{P^B(\tau)B(\tau)}{P(\tau)}$ and $[1 - z(\tau)]\,v(\tau) = m(\tau)$. $c$ and $z$ are the household's control variables, $\xi$ is the co-state variable and $v$ is the state variable with an initial value $v(0) = \frac{V(0)}{P(0)}$. The first order conditions are:

$\frac{\partial H}{\partial c} = \left\{c^{1-\theta}\left[(1-z)\,v\right]^{\theta}\right\}^{-\eta}(1-\theta)\,c^{-\theta}\left[(1-z)\,v\right]^{\theta} - \xi = 0$ or

$$\left\{c^{1-\theta}\left[(1-z)\,v\right]^{\theta}\right\}^{-\eta}(1-\theta)\,c^{-\theta}\left[(1-z)\,v\right]^{\theta} = \xi \tag{51}$$

$\frac{\partial H}{\partial z} = \left\{c^{1-\theta}\left[(1-z)\,v\right]^{\theta}\right\}^{-\eta}\theta c^{1-\theta}\left[(1-z)\,v\right]^{\theta-1}(-v) + \xi\,(rv + \pi v) = 0$ or

$$\left\{c^{1-\theta}\left[(1-z)\,v\right]^{\theta}\right\}^{-\eta}\theta c^{1-\theta}\left[(1-z)\,v\right]^{\theta-1}\frac{1}{r+\pi} = \xi \tag{52}$$

$\dot{\xi} - \rho\xi = -\frac{\partial H}{\partial v} = -\left\{c^{1-\theta}\left[(1-z)\,v\right]^{\theta}\right\}^{-\eta}\theta\left[(1-z)\,v\right]^{\theta-1}(1-z)\,c^{1-\theta} - \xi\left[rz - \pi\,(1-z)\right]$ or

$$\dot{\xi} - \rho\xi = -\left\{c^{1-\theta}\left[(1-z)\,v\right]^{\theta}\right\}^{-\eta}\theta\left[(1-z)\,v\right]^{\theta-1}(1-z)\,c^{1-\theta} - \xi\left[rz - \pi\,(1-z)\right] \qquad (53)$$

Substituting the LHS of (52) for $\xi$ in (51), we have
$\left\{c^{1-\theta}\left[(1-z)\,v\right]^{\theta}\right\}^{-\eta}(1-\theta)\,c^{-\theta}\left[(1-z)\,v\right]^{\theta} = \left\{c^{1-\theta}\left[(1-z)\,v\right]^{\theta}\right\}^{-\eta}\theta c^{1-\theta}\left[(1-z)\,v\right]^{\theta-1}\frac{1}{r+\pi}$
$\Rightarrow r + \pi = \frac{\theta}{1-\theta}c\left[(1-z)\,v\right]^{-1}$
Substituting $m = (1-z)\,v$ we have equation (31) in the text.

Substituting for $\left\{c^{1-\theta}\left[(1-z)\,v\right]^{\theta}\right\}^{-\eta}\theta c^{1-\theta}\left[(1-z)\,v\right]^{\theta-1}$ in the RHS of (53) using (52) yields
$\dot{\xi} - \rho\xi = -\xi\,(r+\pi)\,(1-z) - \xi\left[rz - \pi\,(1-z)\right]$
$\Rightarrow \frac{\dot{\xi}}{\xi} = -\,(r+\pi)\,(1-z) - \left[rz - \pi\,(1-z) - \rho\right]$ or

$$\frac{\dot{\xi}}{\xi} = -\,(r-\rho)$$

Using $m = (1-z)\,v$ in (51), rearranging and taking the derivative with respect to time yields
$(1-\theta)\left\{-\left[\eta\,(1-\theta)+\theta\right]c^{-[\eta(1-\theta)+\theta]-1}m^{\theta(1-\eta)}\dot{c} + \theta\,(1-\eta)\,c^{-[\eta(1-\theta)+\theta]}m^{\theta(1-\eta)-1}\dot{m}\right\} = \dot{\xi}$
Dividing both sides through $\xi$ and taking account of (51) yields an alternative expression for $\frac{\dot{\xi}}{\xi}$:

$$-\left[\eta\,(1-\theta)+\theta\right]\frac{\dot{c}}{c} + \theta\,(1-\eta)\,\frac{\dot{m}}{m} = \frac{\dot{\xi}}{\xi}$$

Equating both expressions for $\dot{\xi}/\xi$ yields equation (32) in the text.

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